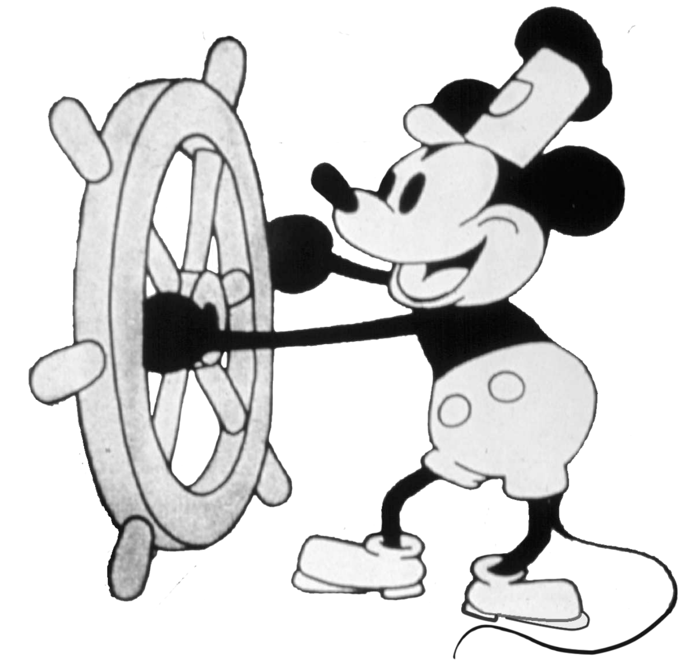


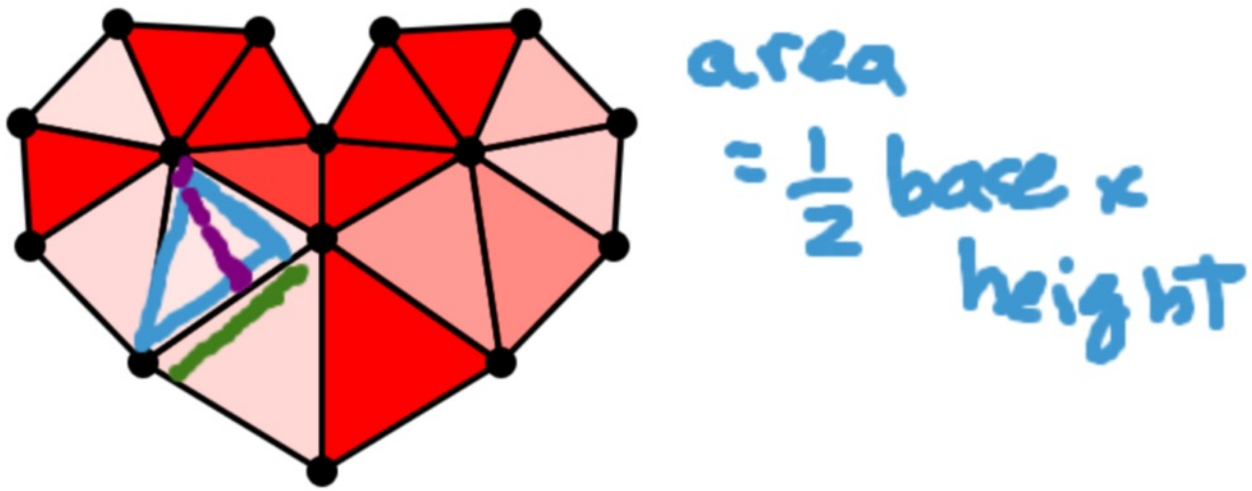
CSCI 461: Computer Graphics

Middlebury College, Fall 2025

Lecture 2A: Linear Algebra (Vectors)

By the end of today's lecture, you will be able to:

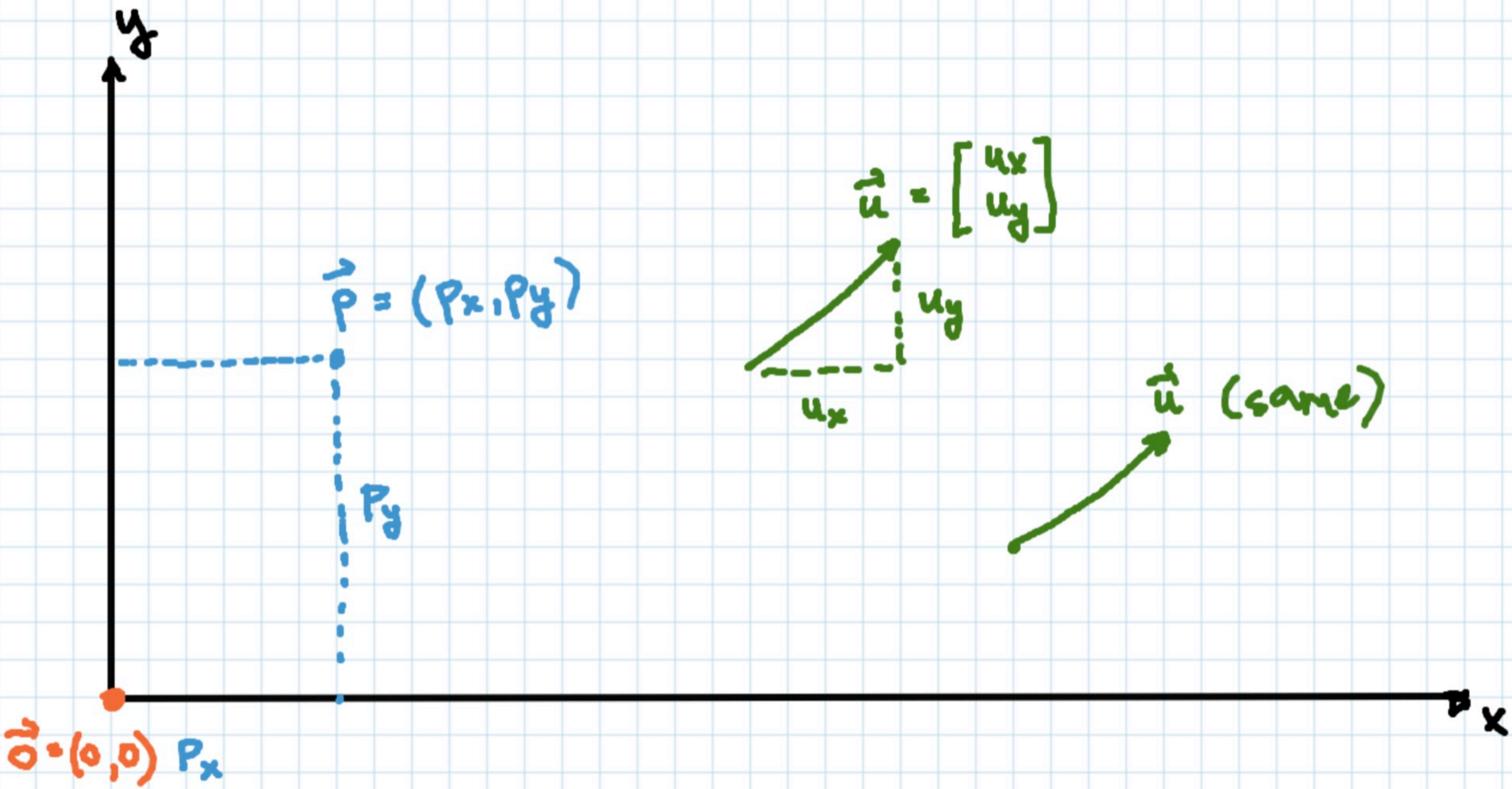
- perform operations on vectors such as **addition, subtraction, scaling**,
- calculate the **length** of a vector and compute **unit vectors**,
- compute the **dot product** and **cross product** of two vectors,
- calculate the **area** of a triangle using the cross product,
- **represent lines and planes** using vector notation,
- use `glMatrix` to do everything mentioned above.



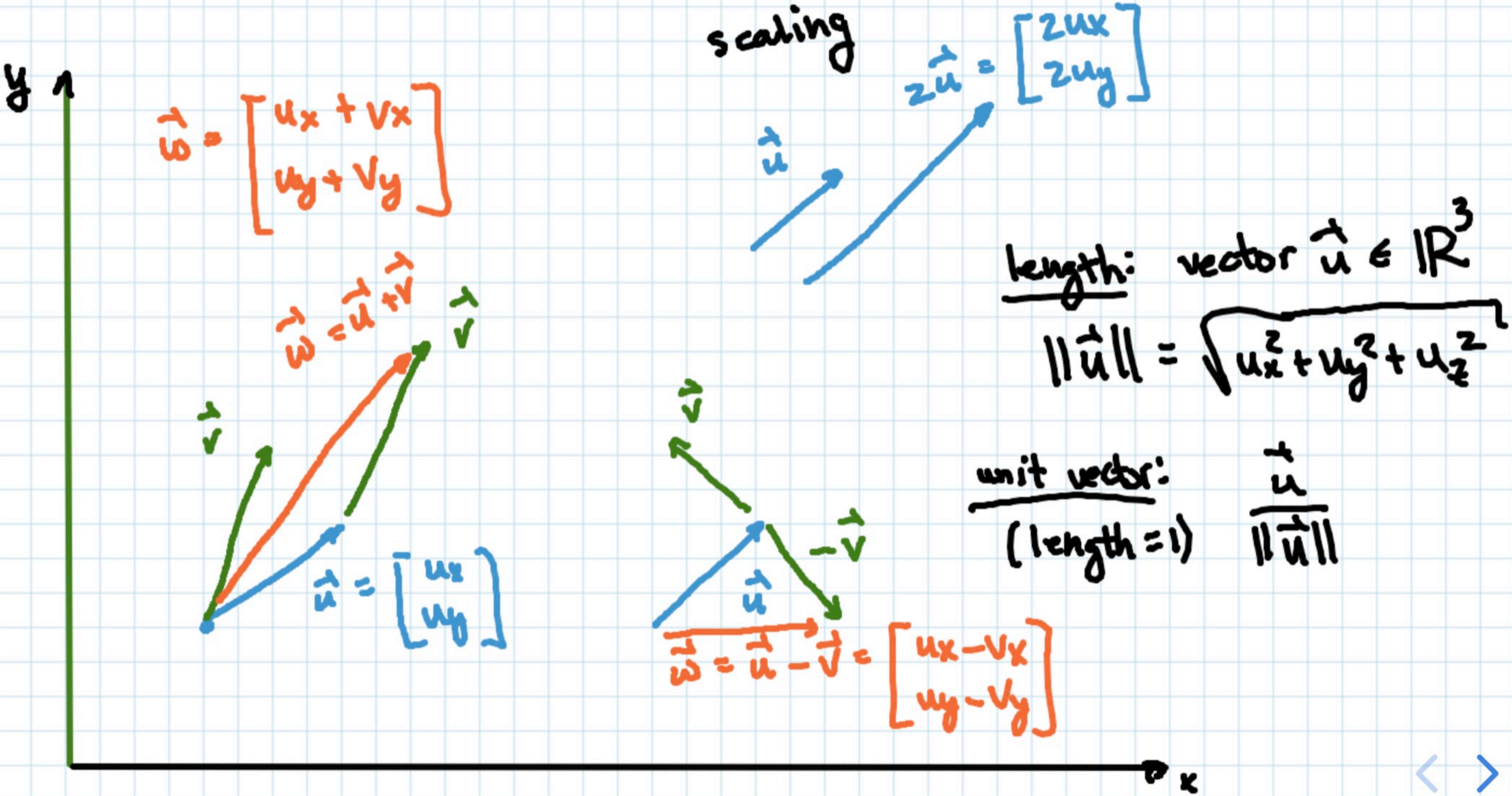
How would you calculate the area of the heart?

Building blocks: points and vectors.

We'll use an arrow on top of symbols to denote points ($\vec{p}$) and vectors ($\vec{u}$).



Vector addition, subtraction, scaling, unit vectors.



Vector addition, subtraction, scaling, unit vectors.

$$\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

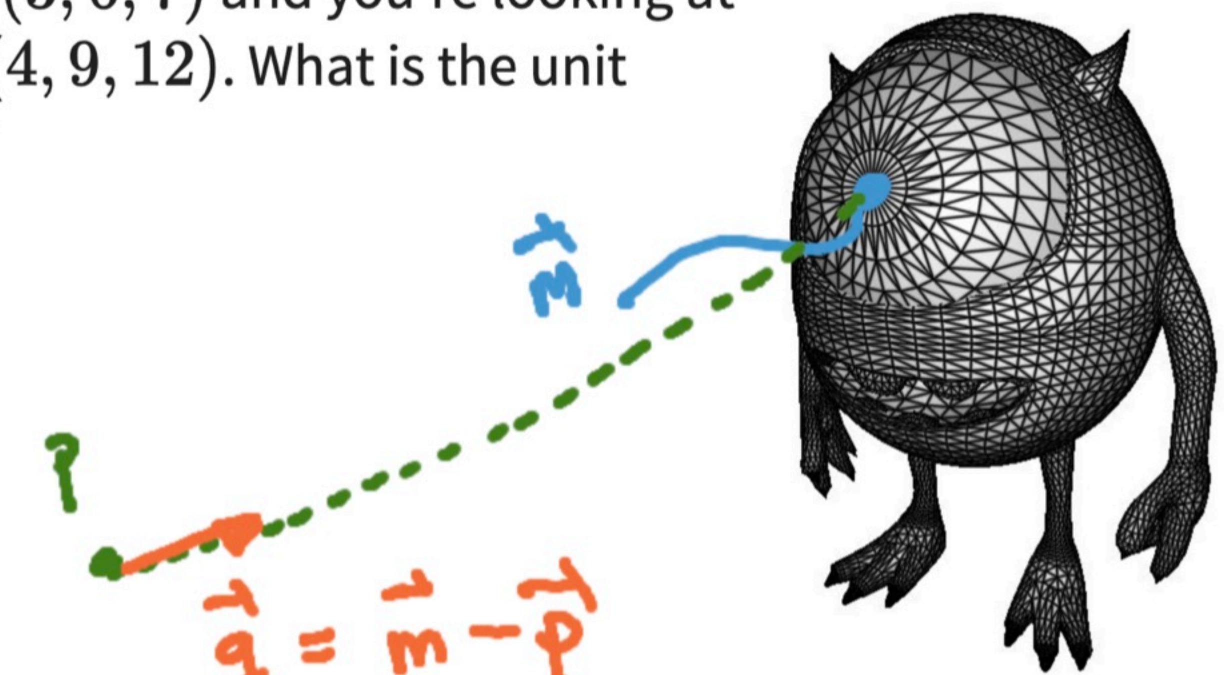
$$\|\vec{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

unit vector has a length of 1

$$\hat{u}_{unit} = \frac{\vec{u}}{\|\vec{u}\|}$$

Example: calculating a unit gaze direction.

Imagine you're standing at a point $\vec{p} = (3, 6, 7)$ and you're looking at Mike Wazowski who is located at $\vec{m} = (4, 9, 12)$. What is the unit vector pointing in the direction of Mike?

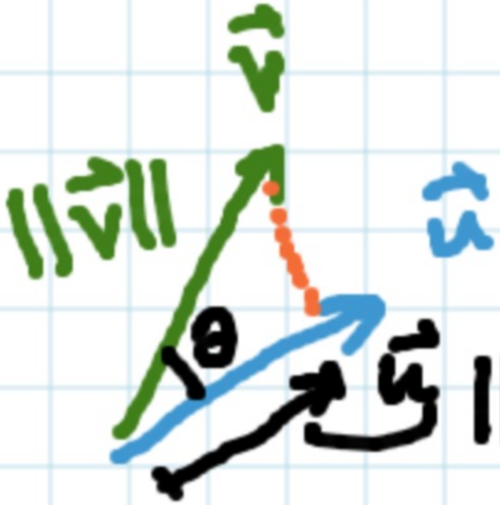


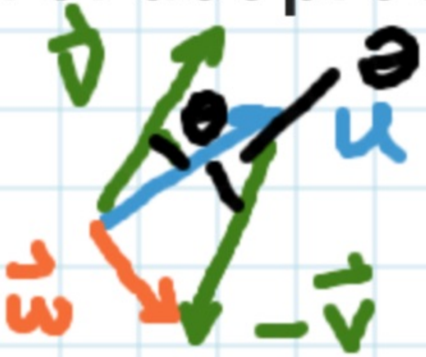
$$\vec{g} = \frac{\vec{m} - \vec{p}}{\|\vec{m} - \vec{p}\|}$$

$$\vec{g} = \frac{\begin{bmatrix}4\\9\\12\end{bmatrix} - \begin{bmatrix}3\\6\\7\end{bmatrix}}{\|\cdots\|} = \frac{\begin{bmatrix}1\\3\\5\end{bmatrix}}{\|\cdots\|} = \frac{\begin{bmatrix}1\\3\\5\end{bmatrix}}{\sqrt{1^2+3^2+5^2}} = \frac{\begin{bmatrix}1\\3\\5\end{bmatrix}}{\sqrt{35}}$$

Derivation of dot product in relation to angles.

$\vec{w} = \vec{u} - \vec{v}$





$\|\vec{u}\|\|\vec{v}\|\cos\theta$

↳ $\|\vec{v}\|\cos\theta\,\vec{u}$

how much of $\vec{v}$ is in the direction of $\vec{u}$

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

(assume 2d)

$$(u_x - v_x)^2 + (u_y - v_y)^2 = u_x^2 + u_y^2 + v_x^2 + v_y^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

expand LHS

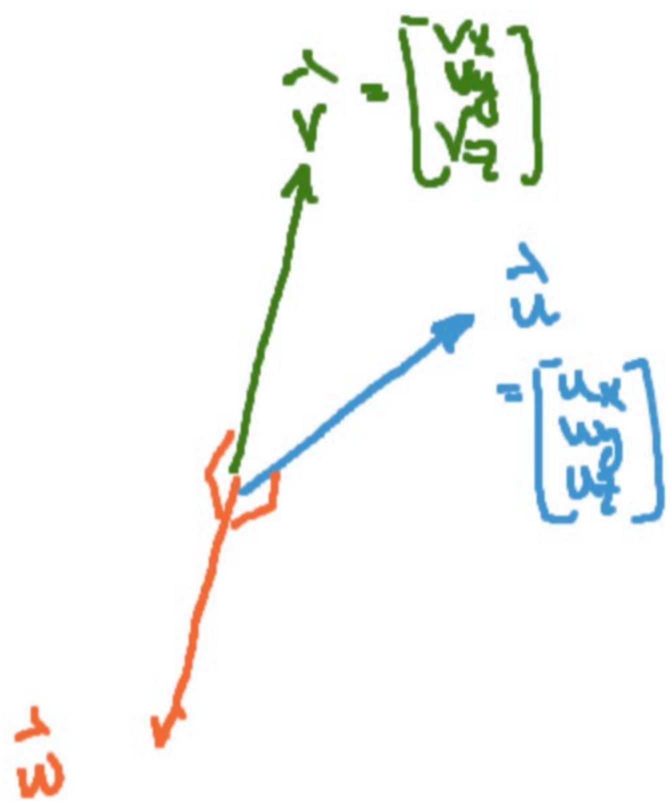
$$u_x^2 + v_x^2 - 2u_x v_x + u_y^2 + v_y^2 - 2u_y v_y = \;\leftarrow$$

$$-2(u_x v_x + u_y v_y) = -2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

dot product

$$\vec{u} \cdot \vec{v} = \sum u_i v_i$$

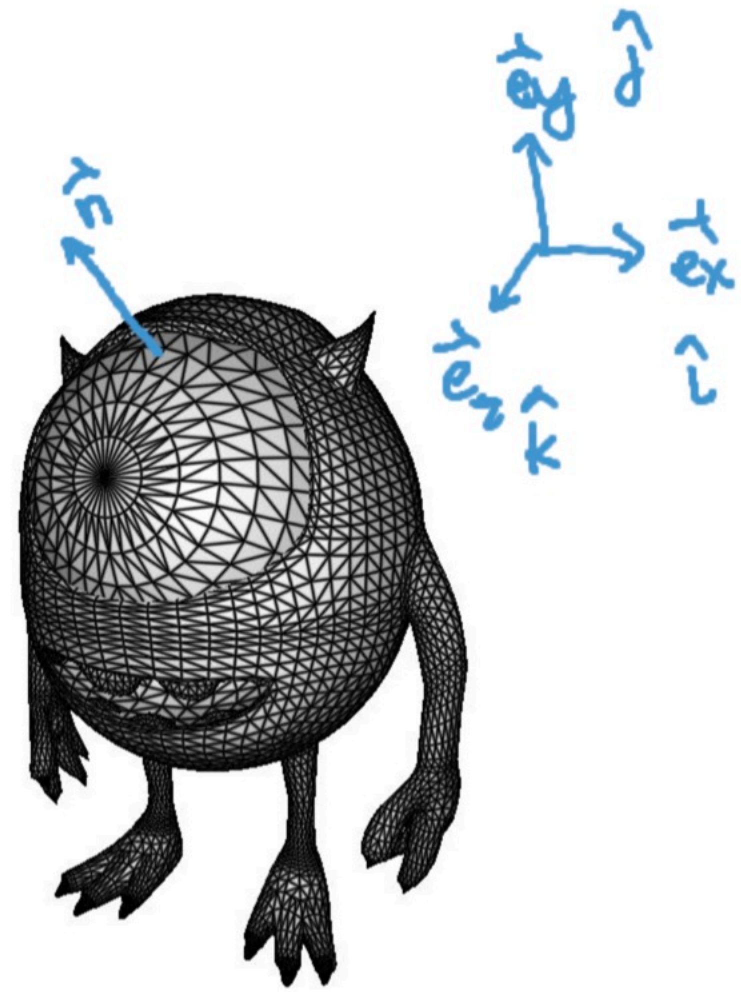
Definition of the cross product.

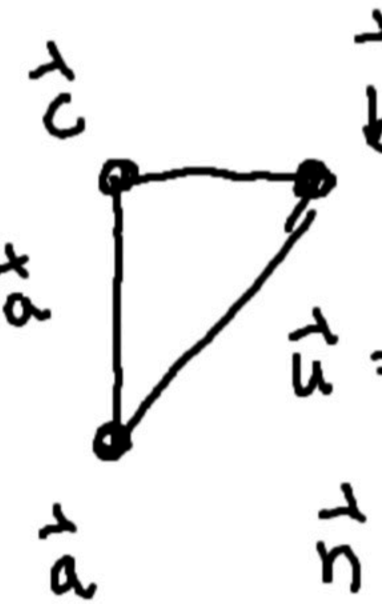


$\perp$ perpendicular.

$$\vec{u} \times \vec{v} = (u_y v_z - u_z v_y)\,\vec{e_x} - (u_x v_z - u_z v_x)\,\vec{e_y} + (u_x v_y - u_y v_x)\,\vec{e_z}$$

order matters!





$\vec{v} = \vec{c} - \vec{a}$

$\vec{u} = \vec{b} - \vec{a}$

$\vec{n} = \vec{u} \times \vec{v}$

area of a triangle $= \frac{1}{2} \|\vec{u} \times \vec{v}\|$

Exercise: show that $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.

$$\vec{u} \times \vec{v} = (u_y v_z - u_z v_y)\vec{e}_x - (u_x v_z - u_z v_x)\vec{e}_y + (u_x v_y - u_y v_x)\vec{e}_z$$

Use:

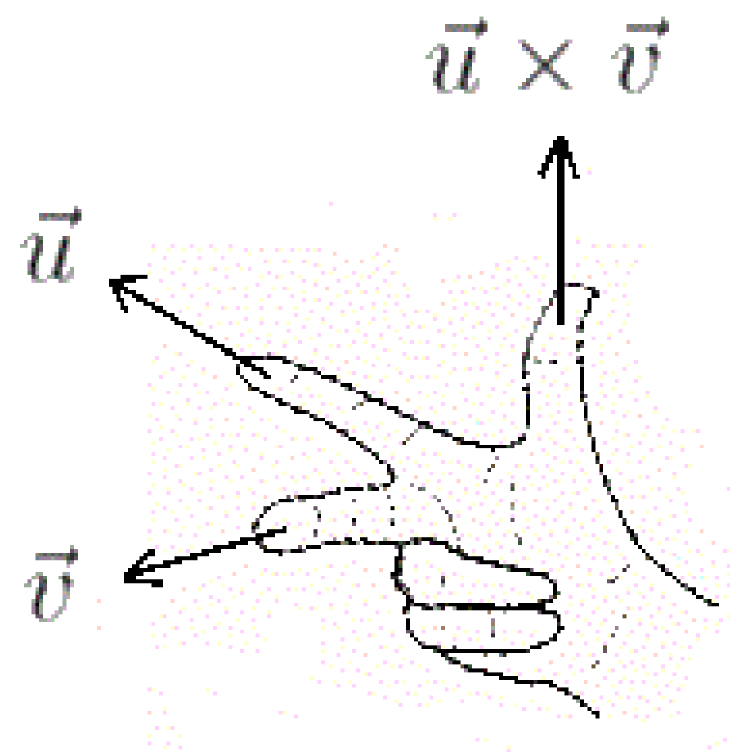
$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solution: first we calculate $\vec{u} \times \vec{v} = \begin{bmatrix} 12 - 15 \\ -(6 - 12) \\ 5 - 8 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$.

Then verify $\vec{u} \cdot (\vec{u} \times \vec{v}) = (1)(-3) + (2)(6) + (3)(-3) = -3 + 12 - 9 = 0$.

Visualizing the cross product.

Using **RIGHT** hand: align index finger with $\vec{u}$, then align middle finger (or all other fingers) with $\vec{v}$ (moving towards your palm). Thumb will point in direction of $\vec{u} \times \vec{v}$.



We will use `glMatrix` for all of our linear algebra calculations.

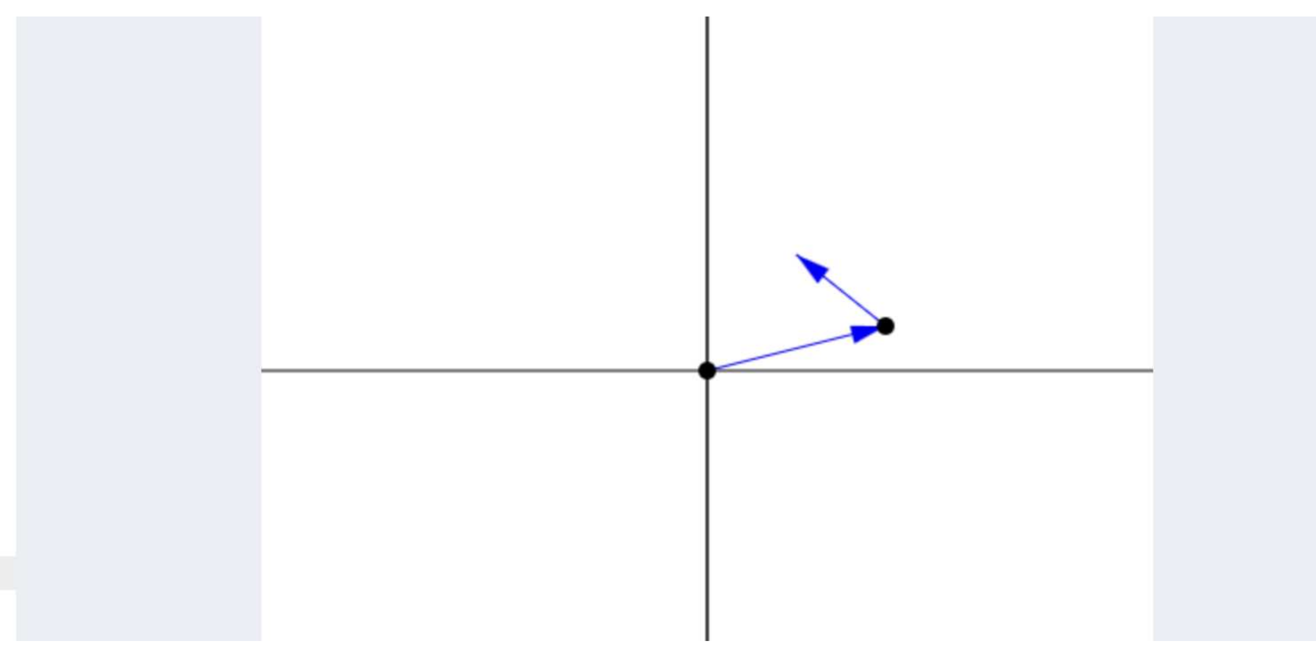
https://glmatrix.net/docs/

FOR FUNCTIONS THAT PRODUCE A VECTOR, THE FIRST ARGUMENT IS THE OUTPUT VECTOR

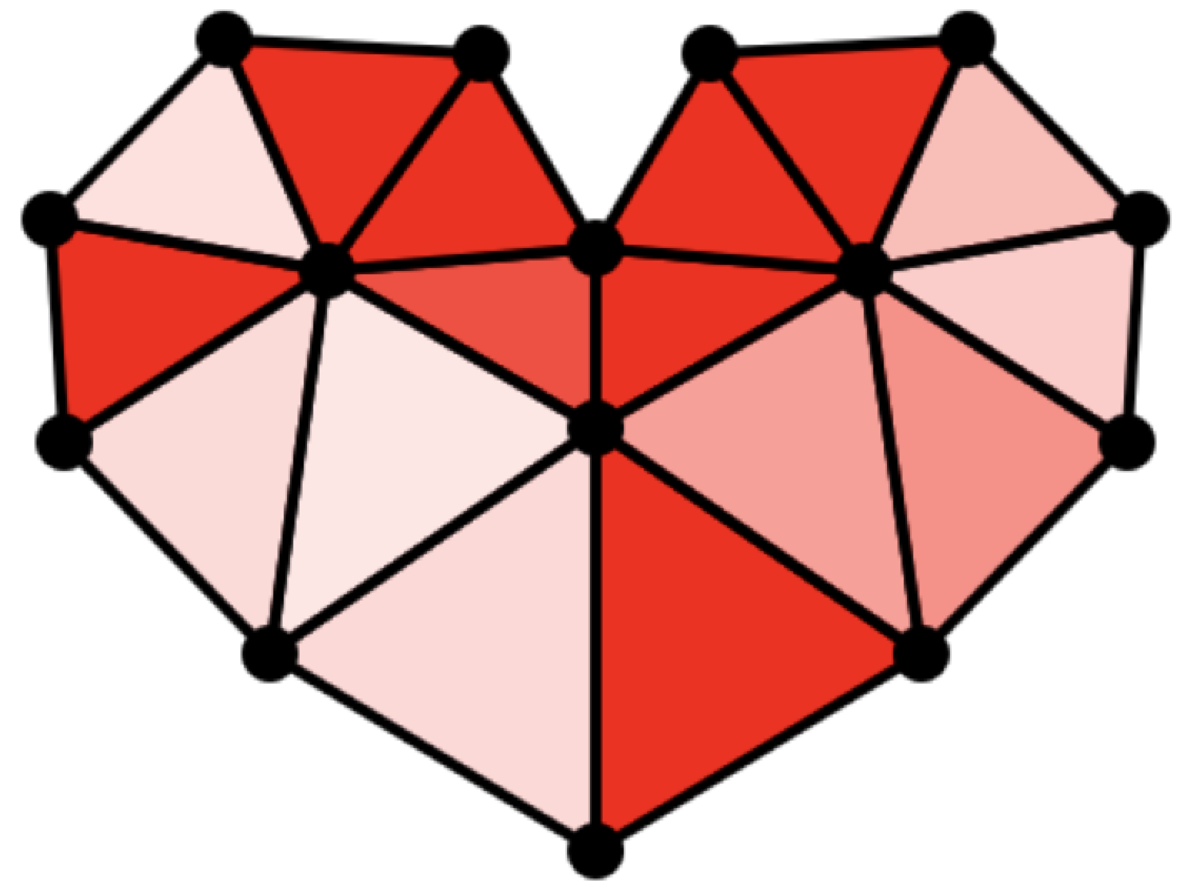
Open a `console` to follow along (go to the Chapter 2A reading page, right-click and **Inspect**).

- Vector addition: `.add(out, u, v)`.
- Vector subtraction: `.subtract(out, u, v)`.
- Vector-scalar multiplication: `.scale(out, u, a)`
- Vector-vector (componentwise) multiplication: `.multiply(out, u, v)`.
- Length of a vector: `.length(u)`.
- Normalize a vector: `.normalize(out, u)`.
- Dot product between two vectors: `.dot(u, v)`.
- Cross product between two vectors: `vec3.cross(out, u, v)`

```

const origin = vec2.fromValues(0, 0);
const u = vec2.fromValues(100, 25);
const v = vec2.fromValues(-50, 40);

const x0 = vec2.add(vec2.create(), origin, u);

// array of points to plot as dots
let points = [origin, x0];

// array of vectors to plot
let vectors = [u, v];

// array of vector tails (optional)
// set an entry to 'undefined' to use the origin
let tails = [undefined, points[1]];

```



Back to our heart example.



- In your groups: split up the work to compute the *total* area of the heart.
- *Hint 1:* use the cross product. Hover over the dots in the notes to get triangle vertex coordinates (doesn't need to be exact).
- *Hint 2:* can you simplify the cross product when $z = 0$ for all points?
- *Hint 3:* look for symmetry...

Summary

- Dot product will be useful for calculating diffuse component of lighting model.
- Cross product useful for calculating perpendicular vectors (and areas).
- Intersections useful for figuring out what we can see (and also what is in a shadow).
- Please try out the `example` assignment on the Setup page.
- **Office hours :** (please come say hi!)
  - Mondays: 10am - 11am
  - Thursdays: 2:30pm - 4:30pm (today)
- Next class: **matrices**.