Middlebury

CSCI 200: Math Foundations of Computing

Spring 2026

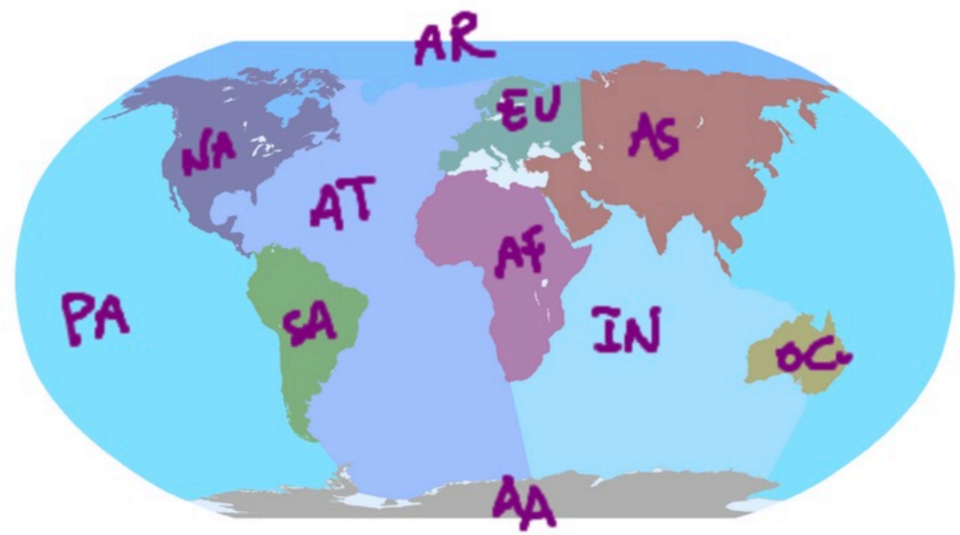
Lecture 13M: Relations

Goals for today:

1. List items in the **Cartesian product** of two sets.
2. Describe (in words) what a **relation** is.
3. Describe three properties of relations: **reflexive, symmetric, transitive**.
4. **Partition** a set into equivalence classes.
5. Show whether or not a relation is an equivalence relation.

The Cartesian Product of two sets: $A \times B$.

$A = \{NA, SA, EU, AF, AS, OC, AA\}$
$B = \{AT, PA, IN, AR\}$



(SA, AR)

Which of these are in the set of ocean-continent pairs that actually border each other?

$A \times B = \{(NA, AT), (NA, PA), (NA, IN), (NA, AR), (SA, AT), (SA, PA), (SA, IN) \ldots\}$

extract subset of cartesian product.

A *relation* R on a set A is a subset of $A \times A$, filtered by some predicate.

A

Let R_1 be the relation on $\mathbb{Z}$: $R_1 = \{(x, y) : x < y\}$. Which of the following are in R_1? Select multiple options.

$\in A$ $\in A$

A. $(1, 2)$
B. $(2, 1)$
C. $(1, 1)$

A

Let R_2 be the relation on $\mathbb{Z}$: $R_2 = \{(x, y) : x \leq y\}$. Which of the following are in R_2? Select multiple options.

A. $(1, 2)$
B. $(2, 1)$
C. $(1, 1)$

In the last example, R_2 satisfied a certain property - what are properties of relations we might care about?

Let A be a set and R is a relation on A, $R \subseteq A \times A$.

- **Reflexive:** $\forall a \in A \quad (a,a) \in R$
- **Symmetric:** $\forall a,b \in A \quad$ if $(a,b) \in R$ then $(b,a) \in R$

  undirected graph adjacencies

  $ab \leq 1$

  $ba \leq 1$
- **Transitive:** $\forall a,b,c \in A$

$$\left.\begin{array}{l}\text{if } (a,b) \in R \\ \text{and } (b,c) \in R\end{array}\right] \rightarrow (a,c) \in R$$

eg. $<$

$a < b$ and $b < c$

then $a < c$.

An *equivalence relation* is a relation that is (1) reflexive, (2) symmetric and (3) transitive.

means $a-b$ is an int. multiple of m

$a-b = k \cdot m$

$k \in \mathbb{Z}.$

Example: Is $R = \{(a,b) : a \equiv b \pmod{m}\}$ an equivalence relation?

yes

reflexive: is $(a,a) \in R$? $\quad a-a = k \cdot m$?

$0 = 0 \cdot m$ ✓

symmetric? if $(a,b) \in R$, is $(b,a) \in R$?

$a-b = k \cdot m \qquad l = -k$

$b-a = -(a-b) = -k \cdot m$

$= l \cdot m$ ✓

transitive? if $(a,b) \in R$ and $(b,c) \in R$ is $(a,c) \in R$?

$a-b = k \cdot m \qquad k, l \in \mathbb{Z}.$

$b-c = l \cdot m$ ✓

is $a-c =$ int. mult. m ? $\quad a-c = a-b+b-c = k \cdot m + l \cdot m$

$= (k+l) \cdot m = n \cdot m$

$n \in \mathbb{Z}.$

$m=4$

partitions $\mathbb{Z}$ into equiv. classes.

M_0	M_1
M_2	M_3

Equivalence relations partition a set into *equivalence classes*.

V_i = *the set of all points in the plane* ($\mathbb{R}^2$) *which are closer to point* i *than to any other point in a list of* n *points.*

$$V_i = \{(x, y) \in \mathbb{R}^2 \colon (x - x_i)^2 + (y - y_i)^2 < (x - x_j)^2 + (y - y_j)^2,\ \forall j \neq i\}.$$

Why is $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : \ a \mid b\}$ not an equivalence relation?

reflexive? $(a,a) \in R$ $\quad a \mid a$ ✓

symmetry? $(a,b) \in R$ is $(b,a) \in R$ $\quad \forall a,b \in A$ $\quad$ (A ↗ $\mathbb{Z}$) $\quad$ ✗

$2 \mid 6$

$6 \nmid 2$ ← "not divides"

People and relations.

Let S be the set of all people. Which of the following are equivalence relations? Why (or why not)? If they are equivalence relations, what are the equivalence classes?

equivalence classes? group of all siblings

1. $R \subseteq S \times S,\ R = \{(a, b) :\ a, b$ have the same parents $\}$

reflexive? ✓ symmetric? ✓ transitive? ✓

2. $R \subseteq S \times S,\ R = \{(a, b) :\ a, b$ share a parent $\}$

not transitive.

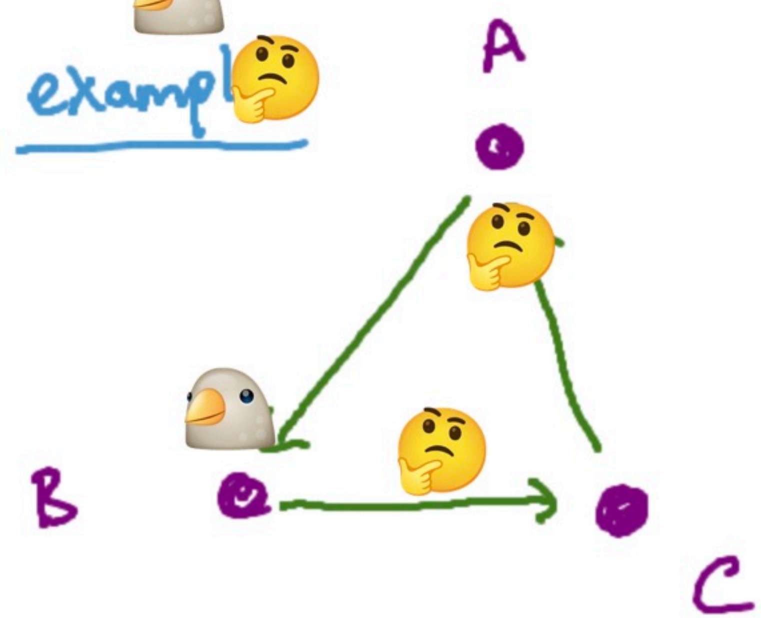
Graphs and relations.

A tournament graph $G = (V, E)$ is a directed graph such that there is either an edge from u to v or an edge from v to u for every distinct pair of vertices u and v. The vertices represent players and an edge $u \to v$ means player u beats player v.

Consider the "beats" relation implied by a tournament graph. Note that a player never beats themselves. Vote on whether the "beats" relation is:

- A. Reflexive
- B. Symmetric
- C. Transitive
- D. None of the above

example

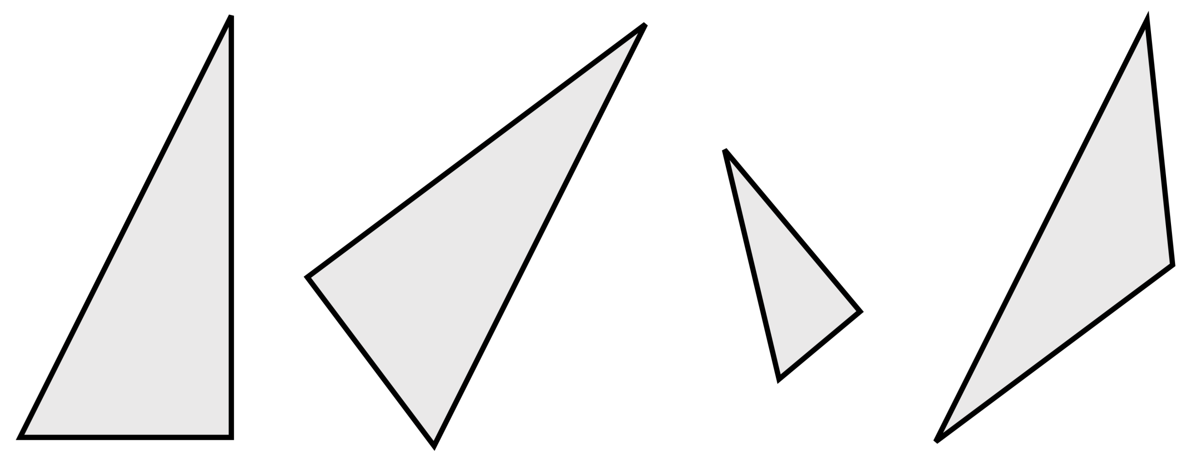


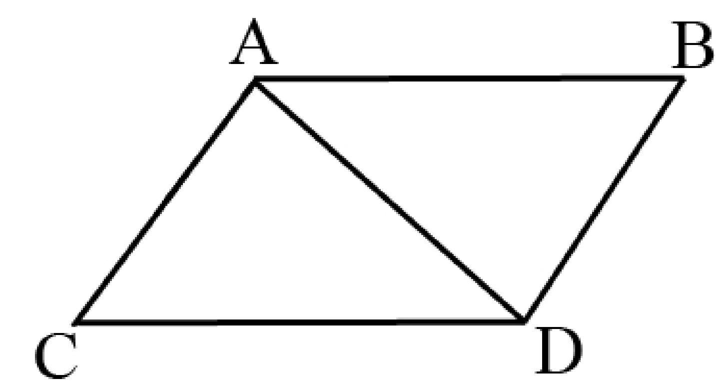
$\forall a, b, c \in A$?

A	B	C	D	E	✋	❤️	👍	👎	😳	🤔	😊	🐦	🐢	🐍
0	0	13	163	0	0	18	3	0	0	63	5	24	0	0

Other applications of equivalence relations.

Geometry and relations: similarity (same shape), congruence (same shape and size), equipollence (lines have same length & direction).





Images from Wikipedia.

Computability theory and relations: two problems A and B are considered *Turing equivalent* if knowing how to solve one of them would allow you to solve the other. Equivalence classes are *Turing degrees* (degree of unsolvability).

The Halting Problem (and our final proof!).

Prove it is not possible to write a program to determine whether some arbitrary computer program p will terminate for some input x.

Proof. We use a proof by contradiction. Suppose such a program h did exist, i.e. it returns "Yes" if p halts with input x or "No" if p runs forever. Now design a new program k and run k on itself (program and input).