Final lab due date: last day (5/11)

Last day for lab feedback: Thursday 5/7

Middlebury

CSCI 200: Math Foundations of Computing

Spring 2026

Lecture 12M: Random Variables

Goals for today:

- Calculate the **expectation** (average) of a **random variable**.
- Apply **linearity of expectation** to calculate the expectation of the sum of several random variables.

Questions this allows us to answer:

- What is the average runtime of an algorithm for a distribution of inputs or random choices made during the algorithm?
- How many Happy Meals do you need to order (on average) to collect all toys?

A *random variable* is actually a *function*.

- Definition (**random variable**): Let S be a sample space. A *random variable* X is a function: $X : S \to \mathbb{R}$

- Definition (**expectation**) (average): Let X be a random variable on a sample space S. The *expectation* of X is: denoted by $E[X]$

  value of rand. var evaluated at s.

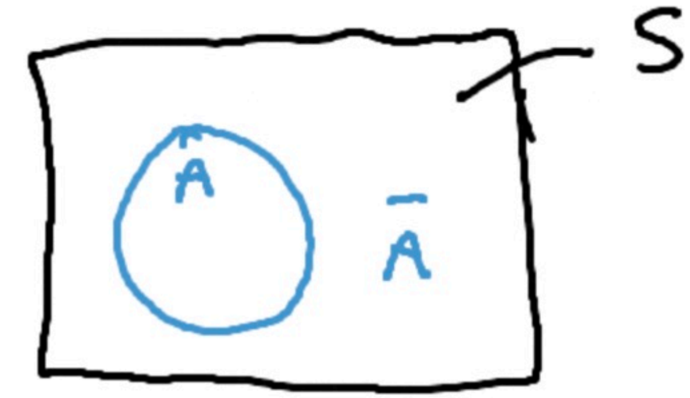
  $$E[X] = \sum_{s \in S} X(s)\, p(s)$$

  prob. of outcome s

- Definition (**indicator random variables**): special type of random variable which is either 0 or 1. $X : S \to \{0, 1\}$

Expectation of an indicator random variable (IRV).

Let A be an event in some sample space S.
Let X be an indicator random variable:



IRV

$$X(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{otherwise} \\ & (s \notin A, \text{ or } s \in \bar{A}) \end{cases}$$

IRV

Theorem: The expected value of X where $X(s) = 1$ for $s \in A$ (0 otherwise) is $E[X] = p(A)$.

$$E[X] = \sum_{s \in S} X(s)\,p(s) = \sum_{s \in A} 1 \cdot p(s) + \cancel{\sum_{s \notin A} 0 \cdot p(s)}^{\,0} = \sum_{s \in A} p(s)$$

$$= p(A)$$

When flipping 4 fair coins, what is the expected (average) number of heads?

2 2 2 2

think of bit strings

$|S| = 2^4 = 16$

Intuition? A: 1, $\boxed{\text{B: 2,}}$ C: 3, D: 4

- Define random variable equal to i if result has i heads.

$$X(s) = \begin{cases} 0 & 0H \\ 1 & 1H \\ 2 & 2H \\ 3 & 3H \\ 4 & 4H \end{cases}$$

$$p(i\,H) = \frac{\#\text{ways to get } iH}{|S|} = \frac{\binom{4}{i}}{16}$$

- Calculate expected value directly.

s is ith outcome

$$E[X] = \sum_{i=1}^{4} X(s)\,p(s) = \frac{0\cdot\binom{4}{0}}{16} + \frac{1\cdot\binom{4}{1}}{16} + \frac{2\cdot\binom{4}{2}}{16} + \frac{3\cdot\binom{4}{3}}{16} + \frac{4\cdot\binom{4}{4}}{16}$$

$$= \frac{4 + 2(6) + 3\cdot 4 + 4}{16} = \frac{32}{16} = 2$$

Linearity of expectation.

Theorem: let X_1 and X_2 be random variables on a sample space S. The expectation of $X_1 + X_2$ is

$$E[X_1 + X_2] = E[X_1] + E[X_2]$$

Let $a,\ b \in \mathbb{R}$. The expectation of $aX + b$ is

$$E[aX + b] = a\,E[X] + b$$

Expected value of sum of two dice.

Suppose you roll 2 fair 6-sided dice. Let X_1 be the outcome of the first die and X_2 the outcome of the second. Show that the expectation of the sum of both dice is 7.

$X = X_1 + X_2$

sum of both dice

want to calculate $E[X]$

$$E[X] = E[X_1 + X_2]$$
$$= E[X_1] + E[X_2] \quad \text{by LOE}$$
$$= \sum_{s\in S} X_1(s)p(s) + \sum_{s\in S} X_2(s)p(s)$$
$$= \sum_{i=1}^{6} i\left(\tfrac{1}{6}\right) + \sum_{i=1}^{6} i\left(\tfrac{1}{6}\right)$$
$$= \frac{1}{6}\left[\frac{6(7)}{2} + \frac{6(7)}{2}\right] = 7$$

arithmetic series

$$\sum_{i=1}^{\square} i = \frac{\square(\square+1)}{2}$$

$$X_1(s) = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases}$$

(same def for $X_2(s)$)

General strategy for applying linearity of expectation.

1. Identify your sample space and random variable of interest, X.

2. Create random variables X_i's which you know how to calculate $E[X_i]$

3. Express random variables X as a weighted sum of indicator random variables.

4. Apply linearity of expectation.

Expected number of heads when flipping 4 coins: now with indicator random variables & linearity of expectation.

$1 \le i \le 4$

- Define sample space as all possible results from flipping 4 coins.
- Create indicator random variable (IRV) for each coin: $X_i(s) = \begin{cases} 1 & \text{if coin } i \text{ is H} \\ 0 & \text{T} \end{cases}$
- Define random variable of interest: $X = \sum_{i=1}^{4} X_i = X_1 + X_2 + X_3 + X_4$
- Apply linearity of expectation

$= p(\text{coin 1 is H})$ since it's an IRV

$$E[X] = E[X_1 + X_2 + X_3 + X_4] = E[X_1] + E[X_2] + E[X_3] + E[X_4]$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{4}{2} = 2$$

Collecting Happy Meal toys.

want $E[M]$

What is the expected number of happy meals (M) you need to buy to collect all n toys (🐕, 🐇, 🐘, 🦒, 🐄)? Example meal sequence to get all 5 toys:

$n=5$ here

$\underbrace{🐕}_{X_1=1}\ \underbrace{🐇}_{X_2=1}\ \underbrace{🐇\ 🐘}_{X_3=2}\ \underbrace{🐕\ 🦒}_{X_4=2}\ \underbrace{🐕\ 🦒\ 🐄}_{X_5=3}$

$M = X_1+X_2+X_3+X_4+X_5$

Let X_i be the # of meals to buy to get the i-th toy. We have $\underbrace{(i-1)}$ different toys so far. Probability of getting i-th toy is (multiple options):

A. $p_i = 1/n$ ✗

B. $p_i = i/n$

C. $\boxed{p_i = (n+1-i)/n}$

D. $\boxed{p_i = \binom{n+1-i}{1} / \binom{n}{1}}$

$n-(i-1) = n+1-i$

$i-1$ toys

X_5

prob. of needing to buy this HM?

$\underline{1}\quad \underline{(1-p_i)}\quad \underline{(1-p_i)^2}$