quiz 9: 16.6

Middlebury

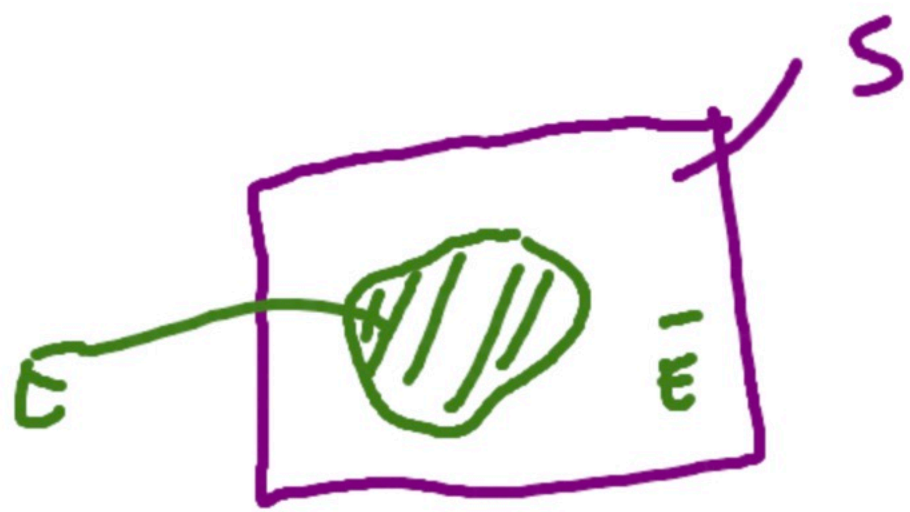
CSCI 200: Math Foundations of Computing

Spring 2026

Lecture 11W: Conditional Probability

Recap of last time (probability definitions).

- State your assumptions!
- Identify sample space (S) and event space (E).
- Draw probability tree to compute outcome probabilities (if possible).
- Outcomes have equal probabilities? $p(E) = \frac{|E|}{|S|}$
- Outcomes have unequal probabilities? $p(E) = \sum_{e \in E} p(e)$
- Probability of E not happening? $p(\overline{E}) = 1 - p(E)$



Goals for today:

- Calculate the probability that an event A occurs, **given** that an event B occurs.
- Determine if two events are **independent**.

I need two volunteers today!

- **Player A:** Do you have *at least* one Ace?
- **Player B:** Do you have the Ace of Hearts?

Everyone else: intuitively, which player has the higher probability of having two Aces. Vote on the course website:

A. Person A has the higher probability.
B. Person B has the higher probability.
C. The probabilities are equal.

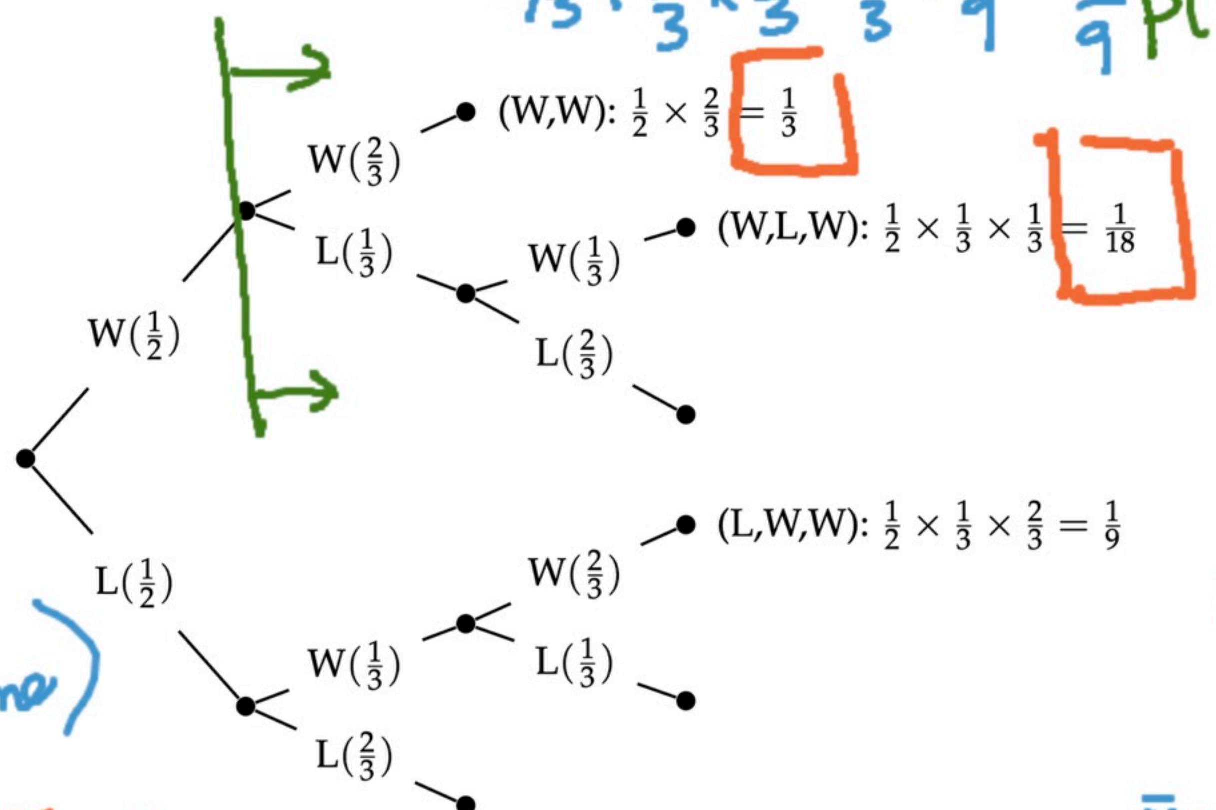
Goal: answer questions like "What is the probability of X *given* Y?"

Example: What is the probability of winning a best 2-out-of-3 series *given* that you win the first game? Assume:

- Each person has a 1/2 chance of winning the first game.
- Chance of winning after a win is 2/3 (so chance of winning after a loss is 1/3).

$2/3 + \frac{1}{3} \times \frac{1}{3} = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$

$P(W_{series}) = \frac{1}{3} + \frac{1}{18} = \frac{7}{18}$



normalize by prob. winning first game

prob(W_{series} given $W_{1st\,game}$) $= \frac{7/18}{1/2} = \boxed{\frac{7}{9}}$ final answer

$\frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \left(\frac{1}{9}\right) = \frac{1}{2}\left[\frac{2}{3} + \frac{1}{9}\right] = \frac{1}{2}\left(\frac{7}{9}\right)$

prob. of winning first game.

Conditional probability and independence.

In the best 2-out-of-3 example, we could have just pruned the probability tree starting from the initial win.

best 2-of-3 series

More generally, we can "filter" the events in which both events of interest occur, and then normalize by the probability of the given event.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\text{prob. of both A and B}}{\text{prob. B}} = \frac{7/18}{1/2} = \frac{7}{9}$$

"given"

independent events: $\underset{\text{(suppose)}}{P(A \mid B) = P(A)} = \frac{P(A \cap B)}{P(B)} \rightarrow P(A \cap B) = P(A) \cdot P(B)$

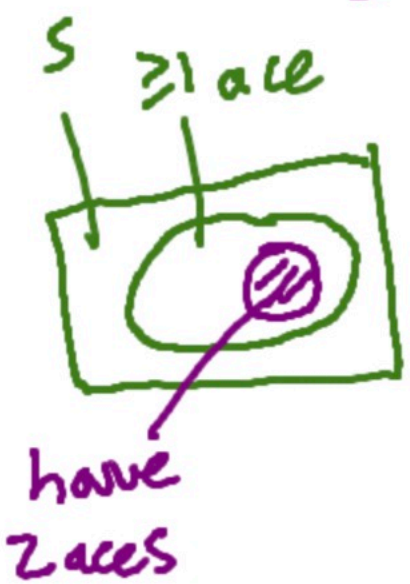
check independence by checking this equation

Back to our 2 Aces example.

Person A: have at least one Ace.

p(have 2 Aces | have at least one Ace)

$= \dfrac{p(\text{have 2 aces} \wedge \text{have at least 1 ace})}{p(\text{have} \geq 1 \text{ ace})}$



$= \dfrac{p(\text{have 2 aces})}{p(\text{have} \geq 1 \text{ ace})}$

$= \dfrac{\#\text{ways to have 2 aces} \,/\, \cancel{\binom{52}{2}}}{\#\text{ways to have} \geq 1 \text{ ace} \,/\, \cancel{\binom{52}{2}}}$

$\binom{4}{2}$ ←

$\binom{52}{2} - \#\text{ways no ace}$
$= \binom{52}{2} - \binom{48}{2}$

$= \dfrac{\binom{4}{2}}{\binom{52}{2} - \binom{48}{2}} = \dfrac{3}{99}$

Person B: have $A\heartsuit$.

p(have 2 Aces | have $A\heartsuit$)

$\dfrac{p(\text{2 aces} \wedge \text{have } A\heartsuit)}{p(\text{have } A\heartsuit)}$

$= \dfrac{\#\text{ways have 2 aces} \wedge A\heartsuit \,/\, \cancel{\binom{52}{2}}}{\#\text{ways to have } A\heartsuit \,/\, \cancel{\binom{52}{2}}}$

$= \dfrac{\binom{3}{1}}{\binom{51}{1}} = \dfrac{3}{51}$

bigger

higher prob. of having 2 aces

Exercise 1: rolling two dice.

1. What is the probability that the sum of both die values adds to 10?
   - A. 1/25
   - B. 10/36
   - C. 1/12
   - D. 1/10

$(1,1), (1,2), \dots, (1,6)$
$(2,1), (2,2), \dots, (2,6)$
$\vdots$
$(6,1), (6,2), \dots, (6,6)$

$p(E) = \frac{3}{36} = \frac{1}{12}$

$|S| = 36$

$|E| = |\{(4,6), (5,5), (6,4)\}|$

2. prob. that the sum is 10 given that at least 1 die is a 5?

how many ways to get  ?

$\{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (1,5), (2,5), (3,5), (4,5), (6,5)\}$

$|E| = 11$

$|S| = 36$

A: sum is 10

B: have 5

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{11/36} = \frac{1}{11}$$

Exercise 2: rolling bit strings of length 4.

A: event in which you randomly create bit strings of length 4 that begin with a 1.

B: event in which you randomly create bit strings of length 4 with an even number of 1's.

Are A and B independent?