Middlebury

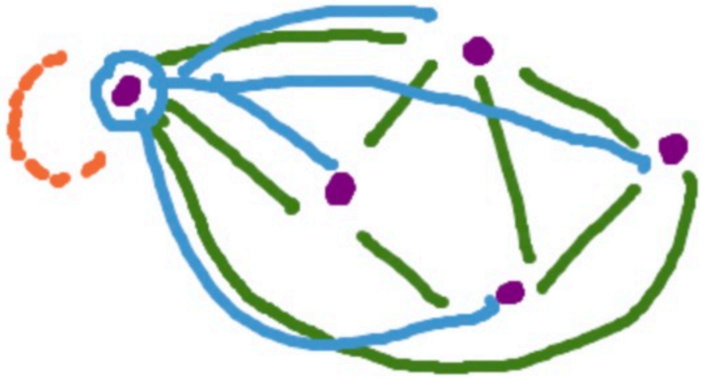
CSCI 200: Math Foundations of Computing

Spring 2026

Lecture 11M: Probability

Counting warmup: how many...

- (a) Simple graphs are there with n vertices?



how many edges? $\binom{n}{2}$ $\quad 2^{\binom{n}{2}}$ graphs

include or exclude a single edge

$n! = n \cdot (n-1) \cdot (n-2)!$ $\quad \binom{n}{2} = \frac{n!}{(n-2)!\,2!}$

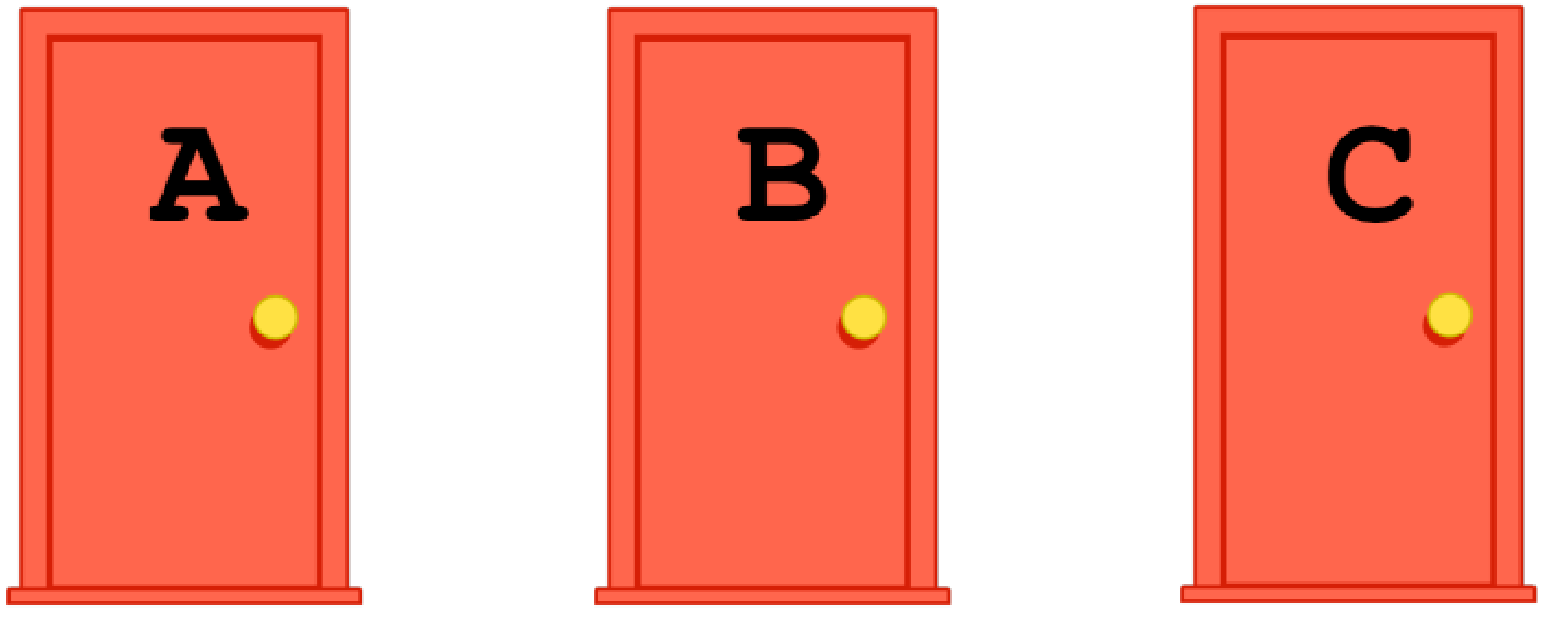
b) graphs with n vertices (allow loops but no multi-edges) $= \frac{n(n-1)}{2}$

$\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n$

$\#\text{edges} = \binom{n}{2} + n$ $\quad \#\text{graphs} = 2^{\binom{n}{2}+n}$

Goals for today:

- Identify the outcomes, events and sample space in a probability problem.
- Use counting techniques to enumerate outcomes and build the sample space.
- Compute the probability of an event by building a probability tree.



I need a volunteer for a game!

Pick the right door and I will tell everyone which problem is on Quiz 9!

Pick the right door and I will tell everyone which problem is on Quiz 9!

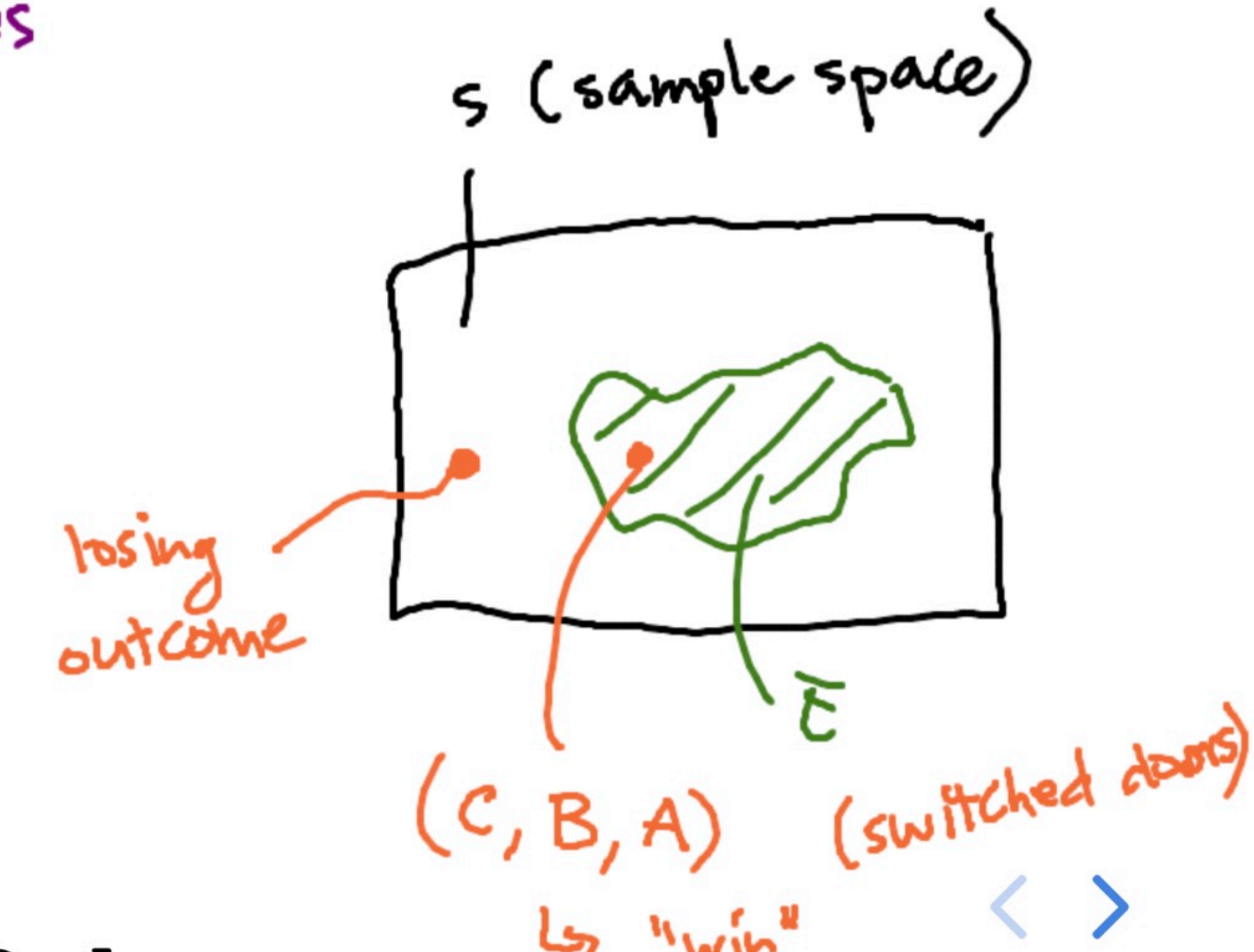
A B C

A few definitions before we can analyze this.

- **Outcome (a.k.a. sample point):** information about some experiment
  e.g.- host picks door
  player picks a door.
  host reveals a door with a goat.

- **Sample space:**
  set of all possible outcomes

- **Event:** subset of sample space



Steps to calculate the probability of an event.

S

1. Find the **sample space**. cardinality $|S|$

E

2. Identity the **event(s)** of interest. cardinality $|E|$

3. Determine the **outcome probabilities** (*multiply* edge weights).

4. Compute **event probability** by *adding* probabilities of outcomes in the event.

all outcomes have equal prob.

$$p(E) = \frac{|E|}{|S|}$$

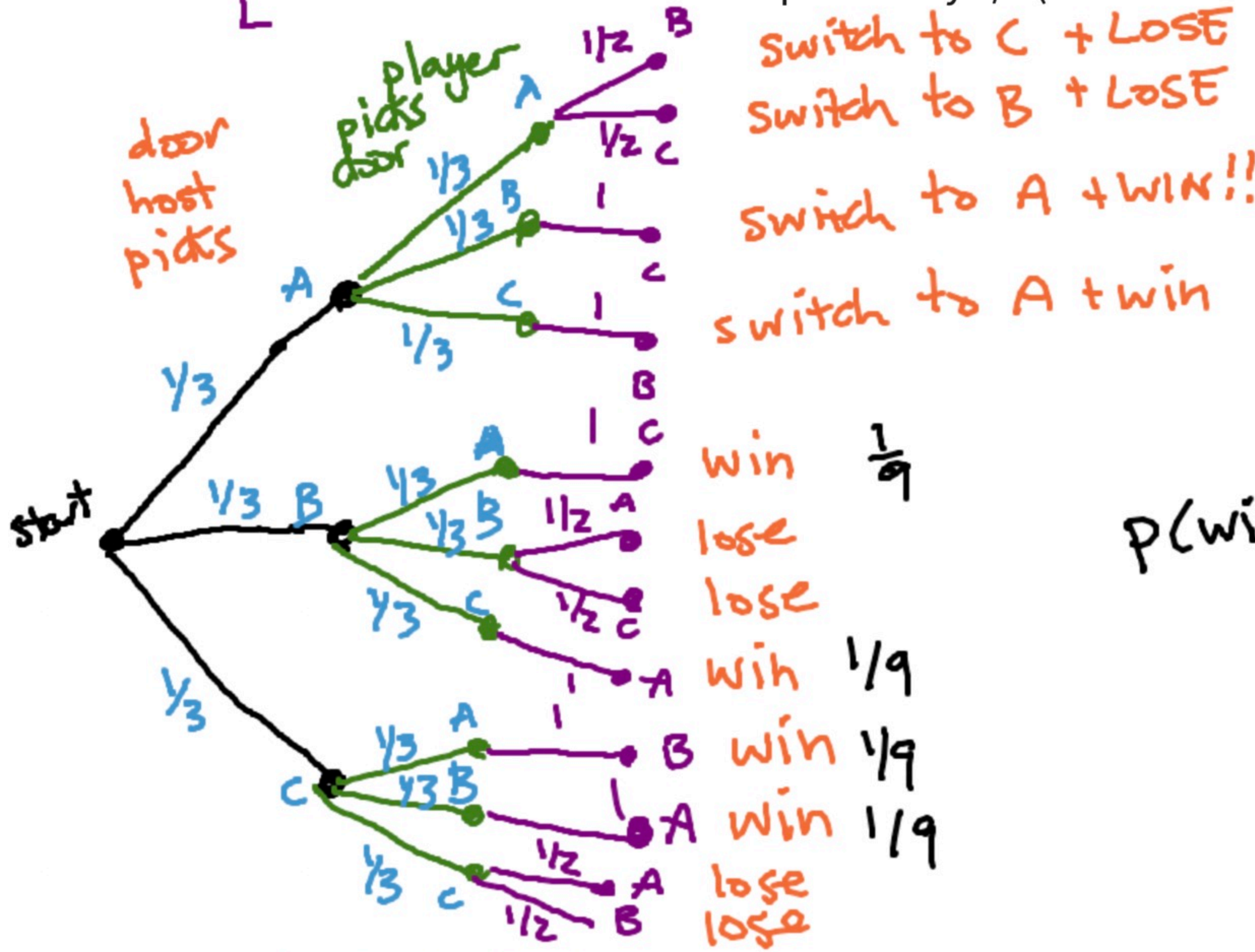
counting problem!

outcomes have unequal prob.

$$p(E) = \sum_{e \in E} p(e)$$

If you always switch doors, what is the probability that you win?

state assumptions
- Host picks a door with probability 1/3.
- Player picks a door with probability 1/3.
- Host reveals other door with probability 1/2 (if there is a choice).



AAB

$$p(\text{lose}): \frac{1}{3} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{18}$$

ABC

$$p(\text{win}) = \frac{1}{3} \times \frac{1}{3} \times 1 = \frac{1}{9}$$

$$p(\text{win})_{ACB} = \frac{1}{3} \times \frac{1}{3} \times 1 = \frac{1}{9}$$

$$p(\text{winning}) = \frac{1}{9} \times 6 = \frac{2}{3}$$

ALWAYS SWITCH DOORS!

Probability of getting a 5-card straight flush?

Straight flush: same suit, values are *consecutive*.
examples: (A, 2, 3, 4, 5), (4, 5, 6, 7, 8), (10, J, Q, K, A), but **not** (Q, K, A, 2, 3).

$\boxed{10}$

values: can start with? A 2 3 4 5 6 7 8 9 10 J Q K

suits: $\boxed{4}$

straight flushes: $4 \times 10 = 40 \to |E|$

hands (5 cards) $= \binom{52}{5}$ ← $|S|$

$P(\text{straight flush}) = \frac{|E|}{|S|} = \frac{40}{\binom{52}{5}}$

More counting practice (cards): how many...

value suits

(same value)

- (a) 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

$\underbrace{13\binom{4}{2}}$

Why not $\cancel{13\binom{4}{2}\binom{48}{3}}$?

$= 13\binom{4}{2}\binom{12}{3}4^3$

4^3

4 options for suit for each.

12 values $\binom{12}{3}$

- (b) 5-card hands have a single 3-of-a-kind and no pair or 4-of-a-kind?

$\underbrace{13\binom{4}{3}}_{\text{triple}}\binom{12}{2}4^2$

- (c) 5-card hands have a single 4-of-a-kind and no pair or 4-of-a-kind?

$13\binom{4}{4}12\cdot 4^1$

More counting practice (cards): how many...

- (a) 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind? $13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3$.
- (b) 5-card hands have a single 3-of-a-kind and no pair or 4-of-a-kind? $13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2$.
- (c) 5-card hands have a single 4-of-a-kind and no pair or 3-of-a-kind? $13 \cdot \binom{4}{4} \cdot \binom{12}{1} \cdot 4^1$.
- (d) 5-card hands (from a deck of 52) contain 2 or more cards with the same value?

(a) + (b) + (c)

2 pairs?

full house?

pair+triple

$\binom{52}{5}$ − # hands all different value

4 4 4 4 4

$\square\ \square\ \square\ \square\ \square$

$\binom{13}{5} 4^5$

(a) + (b) + (c) doesn’t include 5-card hands that have 2 pairs or a pair+triple (full house). We can add these in (see the next slide), or subtract the number of hands where all values are different from the total number of hands.

Python code to analyze previous problem.

```python
from math import comb

n_pairs = 13 * comb(4, 2) * comb(12, 3) * 4 ** 3
n_triples = 13 * comb(4, 3) * comb(12, 2) * 4 ** 2
n_quadruples = 13 * comb(4, 4) * comb(12, 1) * 4 ** 1

n_total = n_pairs + n_triples + n_quadruples
print(f"Number of hands = {n_total}?")

n_hands = comb(52, 5)
n_all_different = comb(13, 5) * 4 ** 5
print(f"Should be {n_hands - n_all_different}")

# two pairs
n_two_pairs = comb(13, 2) * comb(4, 2) * comb(4, 2) * comb(11, 1) * 4

# full house (can also use comb(13, 2) * 2)
n_triple_and_pair = 13 * comb(4, 3) * 12 * comb(4, 2)

n_total += n_two_pairs + n_triple_and_pair
print(f"Number of hands = {n_total}")
```