Middlebury

CSCI 200: Math Foundations of Computing

Spring 2026

Lecture 10W: Combinations & Permutations

Exercise 1: counting bit strings.

Let $T(n)$ be the number of bit strings of length n that **DO NOT** contain two consecutive zeros. $\boxed{\begin{matrix}0\\1\end{matrix}}$ 2   00 × $\boxed{\begin{matrix}10\\01\\11\end{matrix}}$ 3

- Base cases: $T(1) = 2, T(2) = 3.$
- Determine a linear recurrence relation for $T(n)$ in terms of $T(n-1), T(n-2)$.

Let b_n be a bit string of length n without consecutive zeros. Each b_n either ends in 0 or 1.

Note: we can create b_n from b_{n-1}, appending either 0 or 1 (without creating consecutive zeros).

$\overbrace{\underbrace{\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{b_{n-2}}\ \_}_{b_{n-1}}\ \_}^{b_n}$

$T(n-1)$ of these

(1 at end doesn't create consecutive 0's)

case 1: b_n ends with 1 : $b_{n-1}\{\underline{1}\}$ # bit strings without consecutive zeros $= T(n-1)$

case 2: b_n ends with 0 : second to last bit must end with 1

why? consider $b_{n-2}\{\underline{\ }\ \underline{0}\}$, all options are

00× has consecutive zeros

10 ✓ $T(n-2)$ of these

01, 11 included in $T(n-1)$ (case 1)

$\boxed{T(n) = T(n-1) + T(n-2)}$

Exercise 2: counting more bit strings.

$S(n) = 2^n - T(n)$

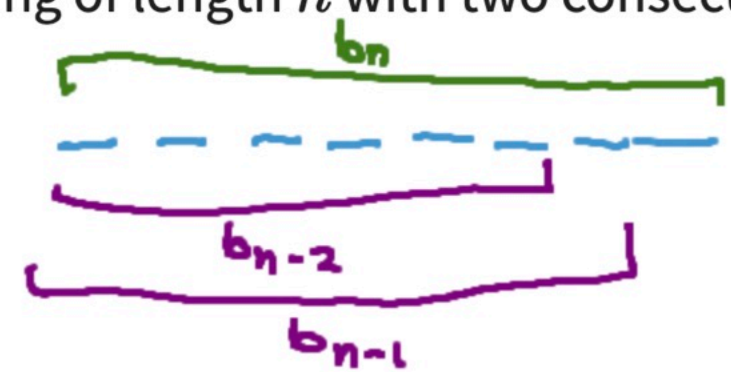
Let $S(n)$ be the number of bit strings of length n that have two consecutive zeros. Consider building a recurrence relation for $S(n)$.

0
1

00 ✓
10
01
11

- Base cases: $S(1) = 0, S(2) = 1$.
- Determine a linear recurrence relation for $S(n)$ in terms of $S(n-1)$, $S(n-2)$ and possibly n.

Let b_n be a bit string of length n with two consecutive zeros. Each b_n either ends in 0 or 1.



case 1: ends with 00, first $n-2$ bits can be anything 2^{n-2} bit strings

case 2: ends with 10, $S(n-2)$ bit strings

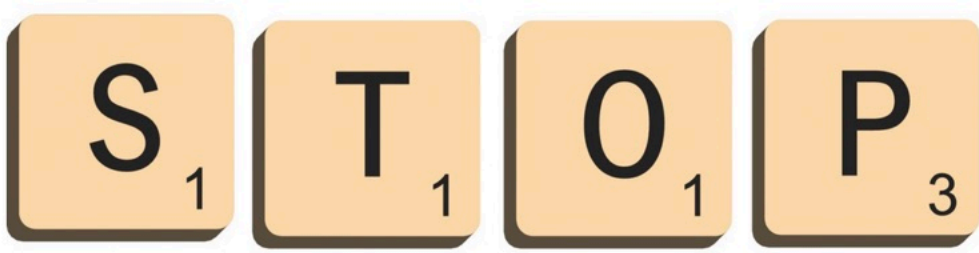
case 3: ends with 01, 11] $S(n-1)$ bit strings

$$S(n) = S(n-1) + S(n-2) + 2^{n-2}$$

Main goal: answer more "in how many ways?" questions.

Today: focus on counting **permutations** and **combinations** of subsets.
Tip: Always ask yourself "does the **order** matter?"

Suppose you're playing Scrabble, and want to search through all arrangements of the following 4 letters to form words. How many arrangements are there?



3 options

2 options

__ __ __ __

4 options

1 option

\# words $= 4!$

(not necessarily real words)

permutation of 4 letters

All possible arrangements:
STOP, STPO, SOPT, SOTP, SPOT, SPTO
TOPS, TOSP, TPOS, TPSO, TSPO, TSOP
OPTS, OPST, OSPT, OSTP, OTSP, OTPS
POTS, POST, PSOT, PSTO, PTOS, PTSO

In Scrabble, we need to form words from a set of 7 letters.

compute :)

POT
TOP

How many 3-letter word arrangments (real word or not) can you make?

Does the order matter? A: Order matters. B: Order does not matter.

total # words: $7 \times 6 \times 5 = \frac{7!}{4!} = P(7,3) = \frac{7!}{(7-3)!}$

options $\underline{7} \times \underline{6} \times \underline{5}$

k-permutation: ordered arrangement of k items from a set of n objects.

$P(n,k) = \frac{n!}{(n-k)!}$ possible permutations.

Asking a different question: in how many ways can we select 3 letters from a set of 7?

$P(7,3) = 210$

repeats:

$E_1 E_2 M U P O T$

POT
TOP

_ _ _

1) no E's $\binom{5}{3}$
2) 1 E $\binom{5}{2}$
3) 2 E $\binom{5}{1}$

25

Will this be less than or greater than previous answer? A: $< P(n,k)$, B: $> P(n,k)$

Does the order matter? A: Order matters, B: Order does not matter.

A k-combination is an **unordered** selection of k distinct items from a set of n objects.

$P(n,k)$ overcounts by $k!$

$$\frac{7!}{(7-3)!\,3!} = \frac{210}{6} = 35$$

$$C(n,k) = \frac{P(n,k)}{k!} = \binom{n}{k}$$

→ "n choose k"

binomial coefficient

$$\binom{n}{k} = \binom{n}{n-k}$$

Problem 1 (lattice paths) In how many ways can we...

Step from (0, 0) to (5, 3)?

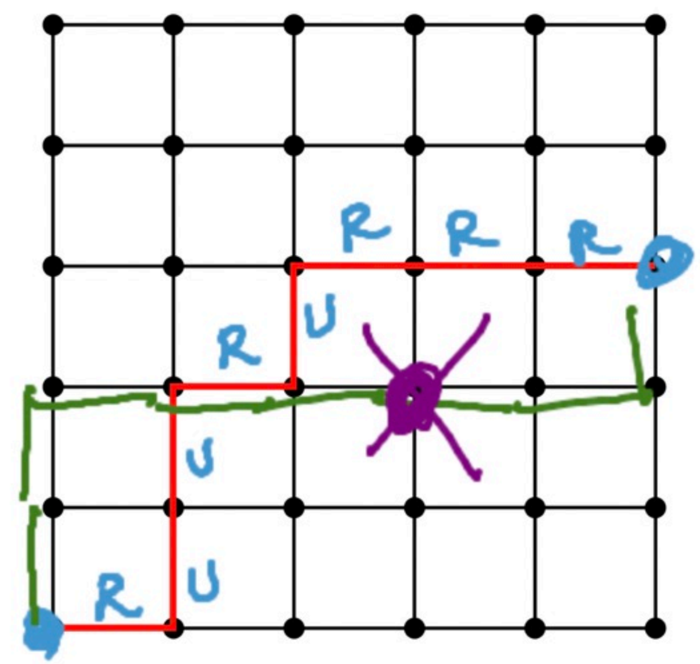
only R or U

RUURU RRR
UURRRRRU

similar to bitstrings

_ _ _ _ _ _ _ _

pick 3 U or 5 R



must be 8 steps

$$\#\text{steps} = \binom{8}{3} = \binom{8}{5} = 56$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Step from (0, 0) to (5, 3) if there is an impassable boulder at (3, 2)?

(0,0) to boulder

(56) (10) 3

$$\binom{8}{3} - \binom{5}{2}\binom{3}{2} = \frac{8!}{3!\,5!} - \frac{5!}{2!\,3!}\cdot\frac{3!}{2!\,1!} = 26$$

(0,0) → (5,3)

(boulder) → (5,3)

Problem 2 (stars and bars): in how many ways can...

10 identical candies be shared between two people?

x : candies #1 gets

y : candies #2 gets

$x + y = n$

$n = 10$

n+1 locations to pick from

picking 1 location for bar.

$$\binom{11}{1} = \frac{11!}{1!\,10!} = 11$$