office hours change: (this week)

Thursday 1-3 pm → Tuesday 1-3 pm

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Middlebury

CSCI 200: Math Foundations of Computing

Spring 2026

Lecture 9M: Graph Coloring

Goals for today:

1. Describe the concept of graph coloring and why it's useful.
2. Prove that a graph with a maximum degree of k can be colored with $k + 1$ colors.
3. Implement a graph coloring algorithm!

Can you come up with an exam schedule for these 5 courses?
(X denotes an overlap in enrollment for two courses)

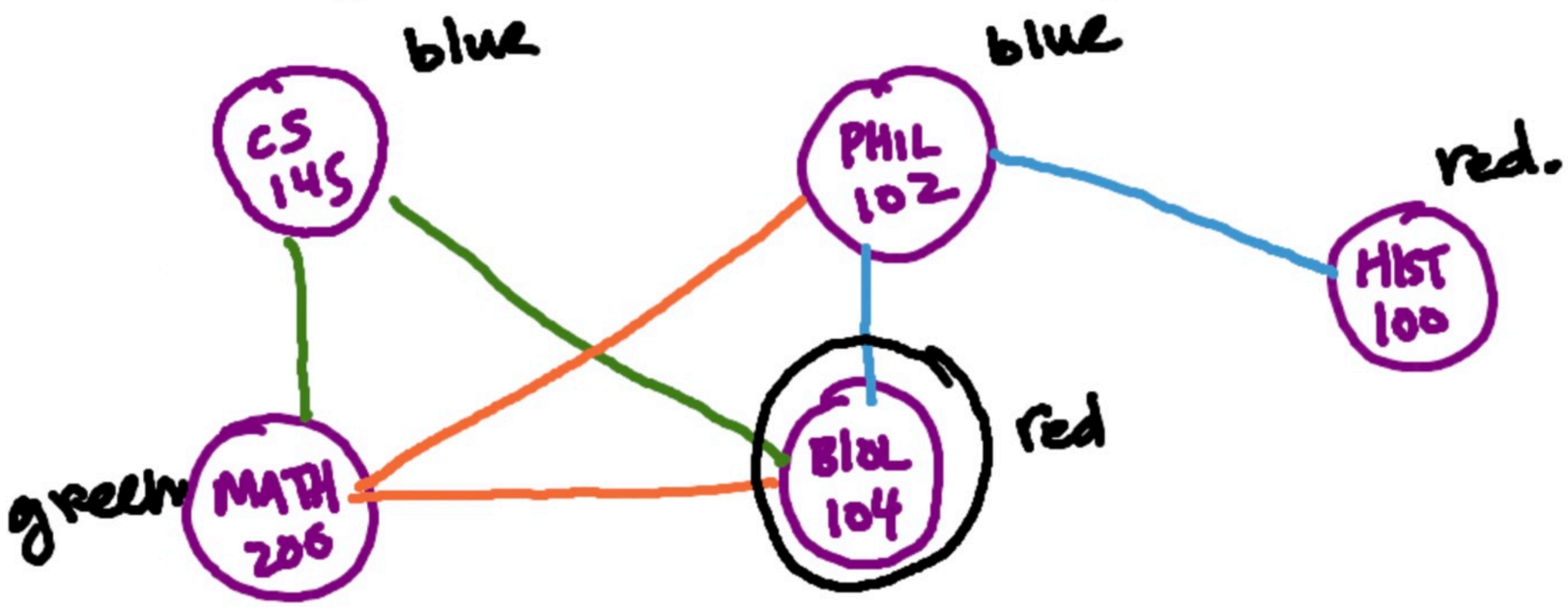
	CSCI 145	MATH 200	PHIL 102	BIOL 104	HIST 100
CSCI 145		X		X	
MATH 200	X		X	X	
PHIL 102		X		X	X
BIOL 104	X	X	X		
HIST 100			X		

Try it out and then vote! **A:** 1 color, **B:** 2 colors, **C:** 3 colors, **D:** 4 colors, **E:** 5 colors.

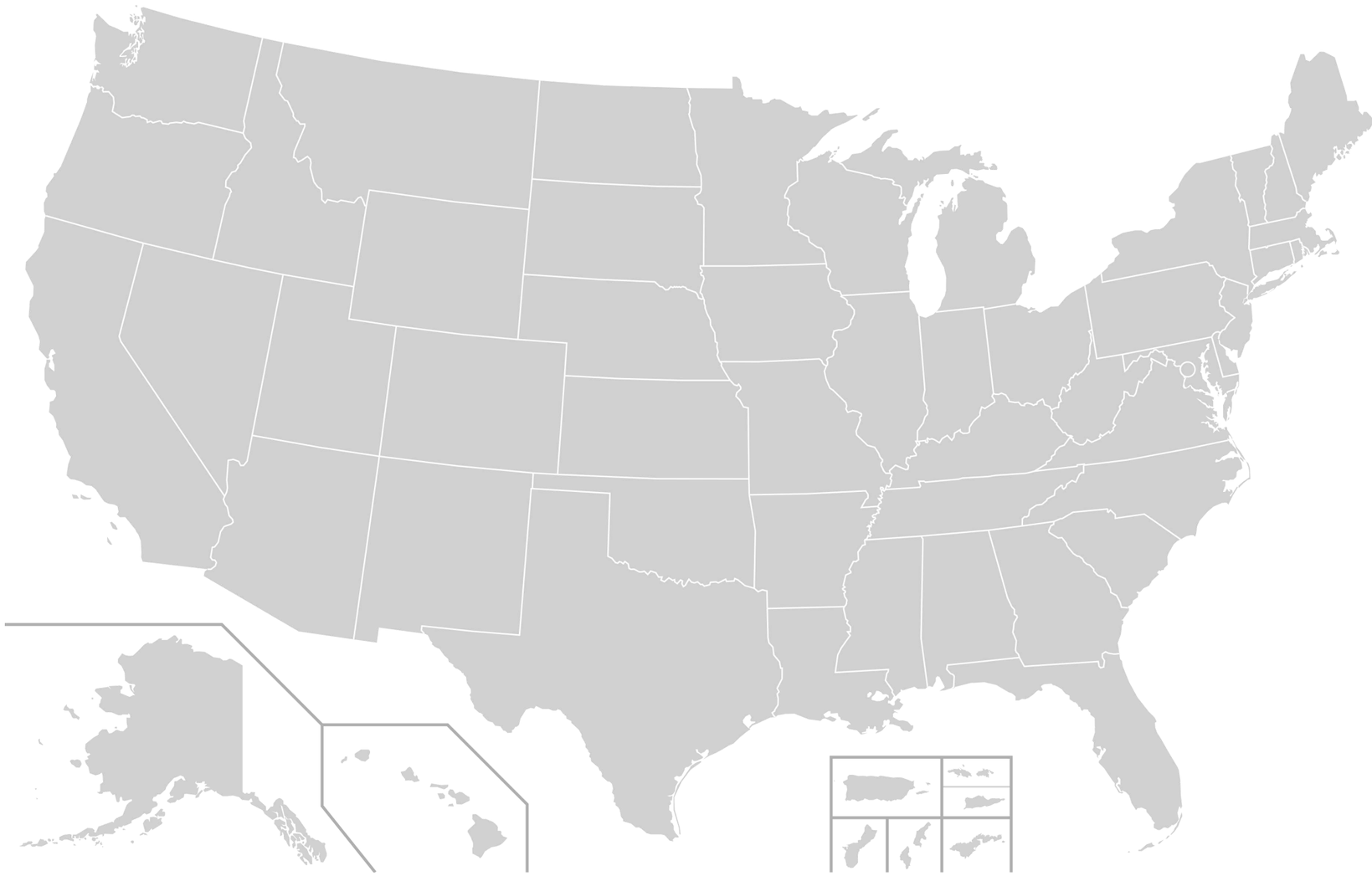
Let's interpret this as a graph:

	CSCI 145	MATH 200	PHIL 102	BIOL 104	HIST 100
CSCI 145		X		X	
MATH 200	X		X	X	
PHIL 102		X		X	X
BIOL 104	X	X	X		
HIST 100			X		

Goal: assign colors to nodes so that two adjacent nodes have a different color.



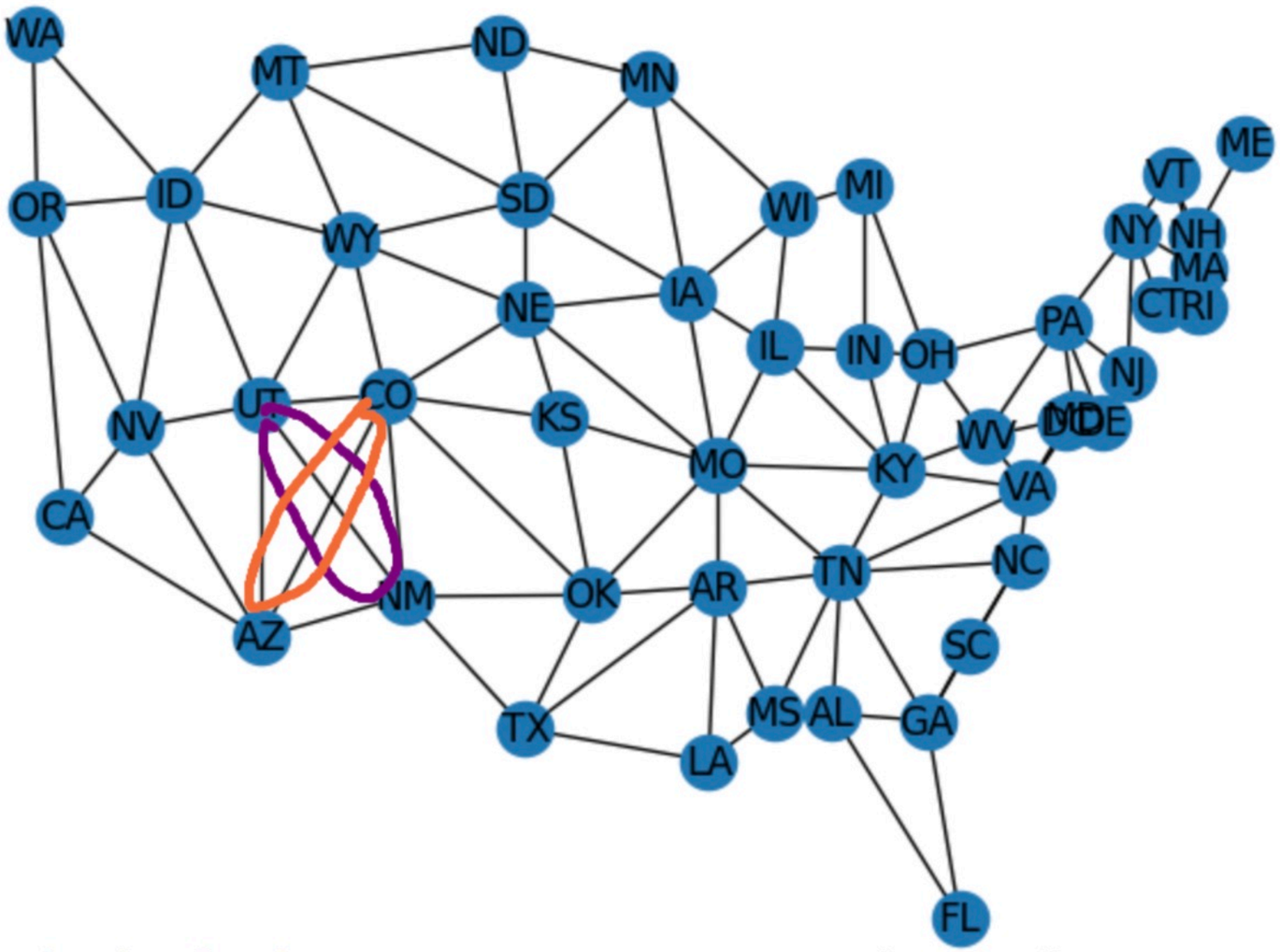
Also useful for coloring maps!

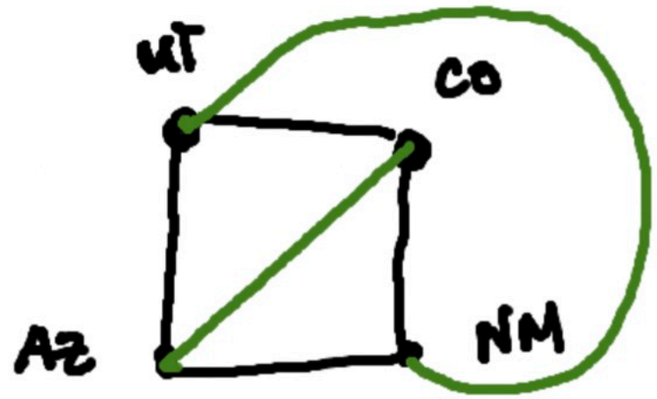


The Four Color Theorem: A *planar* graph can be colored with 4 colors.
A **planar graph** is a graph in which the vertices can be *embedded* in the plane such that no edges cross (edges can be curved though).

What if we include the diagonal adjacencies at the Four Corners?

- **Nodes** are states and there is an **edge** between states that share a border (adjacent).





An upper bound on the number of colors needed.

Theorem: A graph $G = (V, E)$ with $n = |V|$ vertices and a maximum degree of k can be colored with $k + 1$ colors.

Proof: We use a proof by induction on the # of vertices n. Let the induction hypothesis be the predicate p(n):

"*A graph with n vertices and maximum degree k can be colored with $k + 1$ colors.*"

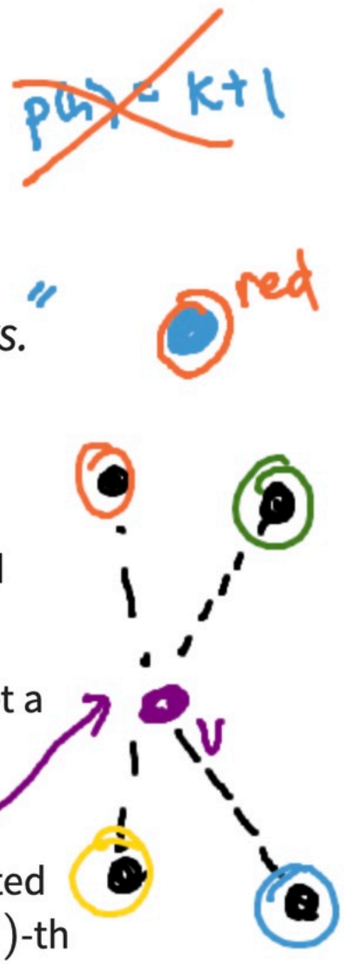
- **Base case:** For n=1, we have 0 edges, so the maximum degree is 0. This single node/vertex can be assigned a single color, which is the maximum degree (0) + 1.

- **Inductive step:** Assume p(n) is true. We need to show a graph with n+1 vertices and maximum degree k can be colored with k+1 colors.

  Consider a graph G with k or k+1? n+1 vertices and maximum degree k. Remove a vertex v from G to get a n-vertex graph G'. The maximum degree of G' is also k which can be colored with $k + 1$ colors by our assumption.

  Let's add v back in. Since the degree of v will be at most k, suppose (in the worst case), all vertices connected to v have a different color, which means there are k colors we cannot pick from. Therefore, pick the $(k + 1)$-th color for v, which shows that G' can be colored with $k + 1$ colors.

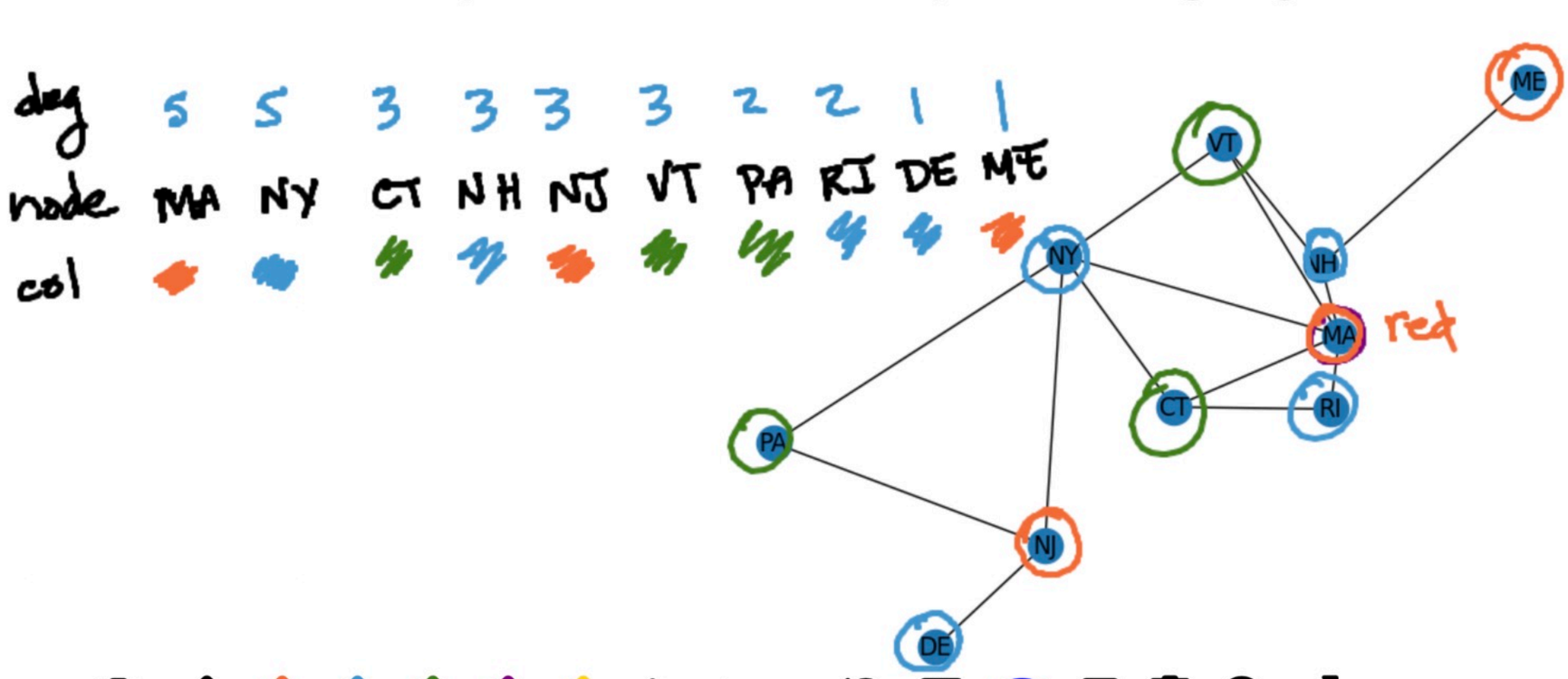
Therefore, by induction on n, a graph with maximum degree k can be colored with $k + 1$ colors. □



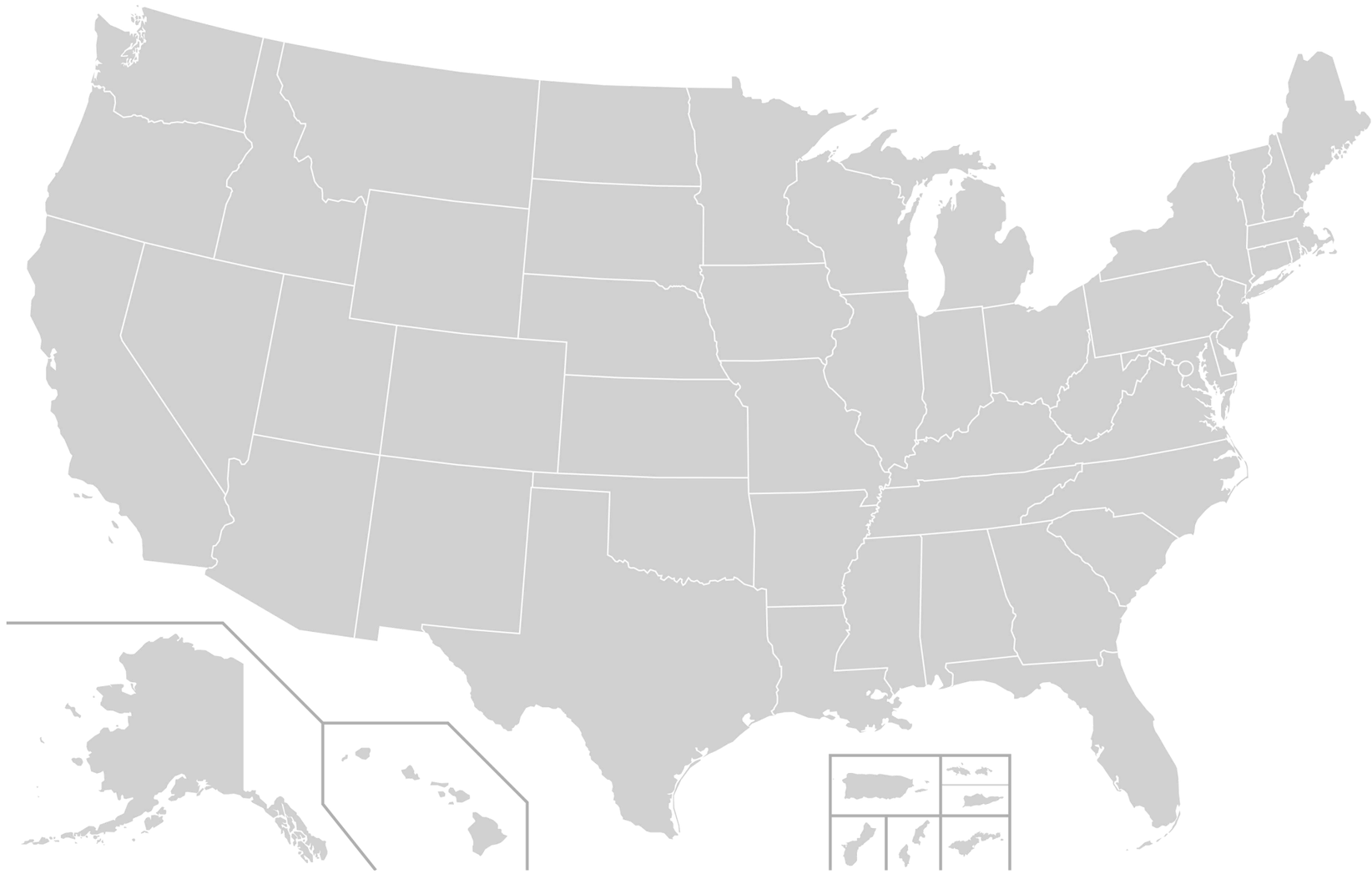
We might be able to use fewer colors, but this bound helps us develop an algorithm for coloring the nodes of a graph.

1. Determine the maximum degree k of the graph. We know we need at most $k + 1$ colors.
2. Iterate through all possible colors i ($0 \leq i < k + 1$).
    - Iterate through all nodes with an *unassigned color*.
        - Check if color i can be used for unassigned node u: if any node v adjacent to u is assigned color i, then we cannot use color i for node u.

It also helps to sort the nodes by decreasing degree beforehand. Try it out!



Exercise: complete the code in the notebook (see "exercise" link in the row for today's class) to assign a color to each state.

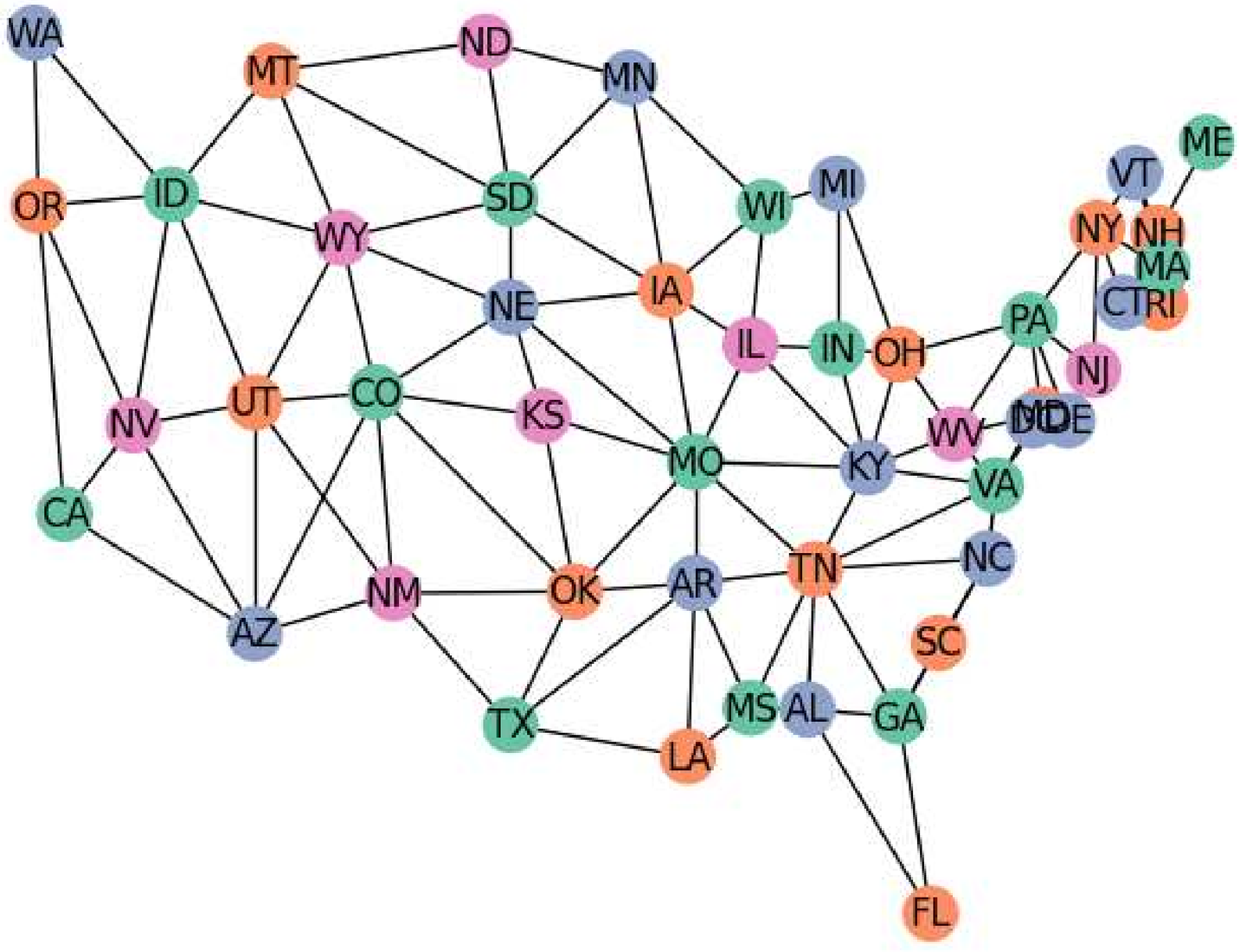


Possible **Python** code for our graph coloring algorithm.

```python
def color_graph(adj):
  nodes = list(adj.keys())
  colors = {}
  degrees = {node: len(adj[node]) for node in adj}
  nodes = sorted(nodes, key=lambda u: degrees[u], reverse=True)

  k = len(adj[max(adj, key=lambda u: len(adj[u]))])
  for i in range(k + 1):
    if len(colors) == len(nodes):
      print(f"Completed using {i} colors (of max = {k + 1}).")
      break
    for u in nodes:
      if u in colors:
        continue
      color_ok = True
      for v in adj[u]:
        if v in colors and colors[v] == i:
          color_ok = False
          break
      if color_ok:
        colors[u] = i
  return colors
```

The result!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