Middlebury

CSCI 200: Math Foundations of Computing

Spring 1996

Lecture 7W: Graphs

Which of the following statements are true?

✓ A. Regular induction might use ~~require proving~~ multiple base cases.

✗ B. Strong induction always requires proving multiple base cases.

C. A and B are true.

↳ chocolate example from Monday only required one base case

→ usually, one base case is enough for regular induction.

Sometimes, multiple base cases are useful.

e.g. Prove any $n^{¢} \geq 8$ can be made with $3^{¢}$ and $5^{¢}$ stamps.

it ind. step uses: "Assume $n \geq 9$ can be made from $3^{¢}$ and $5^{¢}$ stamps".

case 1: $\boxed{\text{no } 5^{¢} \text{ stamps}}$ (in n¢ pile) → must have $\geq 3 \times 3^{¢}$. Remove $3 \times 3^{¢}$, Add $2 \times 5^{¢}$] get $(n+1)^{¢}$

case 2: $\boxed{\text{have } 5^{¢} \text{ stamp}}$ (in $n^{¢}$ pile) → Remove $1 \times 5^{¢}$, Add $2 \times 3^{¢}$] get $(n+1)^{¢}$

But ind. step assumes $n \geq 9$, so we should prove $n=8$ $(1 \times 3^{¢} + 1 \times 5^{¢})$ and $n=9$ $(3 \times 3^{¢})$ as base cases.

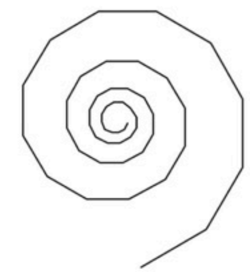
→ But it is possible to prove this using one base case (similar to lemma with coin example)

Proving algorithm correctness (or a property).

Prove that the algorithm below draws a total length of $d(1-\alpha^n)/(1-\alpha)$. $= d\left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)$

Recursive Spiral with the `turtle`

```
1 import turtle
2
3 def spiral(n, d, alpha):
4     if n == 0:
5         return
6     turtle.left(30)
7     turtle.forward(d)
8     spiral(n - 1, alpha * d, alpha)
```



Sample output of `spiral(50, 50, 0.95)`.

$d + \alpha d\left(\frac{1-\alpha^n}{1-\alpha}\right)$

$= d\left[1 + \frac{\alpha - \alpha^{n+1}}{1-\alpha}\right] = d\left(\frac{1-\cancel{\alpha}+\cancel{\alpha}-\alpha^{n+1}}{1-\alpha}\right)$

Proof. We use a proof by induction on the number of generations n. Let the induction hypothesis be $p(n)$: "`spiral(n, d, alpha)` draws a total length of $d\frac{1-\alpha^n}{1-\alpha}$".

- **Base case:** Our base case is at $n = 0$, in which case nothing is drawn. Line 5 correctly draws nothing at $n = 0$, which agrees with $p(0)$ since $d\frac{1-\alpha^0}{1-\alpha} = \frac{0}{1-\alpha} = 0$.
- **Inductive step:** Assume $p(n)$ is true. That is, the total length drawn by `spiral(n, d, alpha)` $= d\frac{1-\alpha^n}{1-\alpha}$. We need to show spiral(n+1, d, alpha) draws $d\frac{(1-\alpha^{n+1})}{1-\alpha}$.

SKETCH

Line 7: draws d.

Line 8: draws spiral(n+1-1, α·d, α) = spiral(n, $\underbrace{\alpha d}$ "d", α) draws $\alpha d\left(\frac{1-\alpha^n}{1-\alpha}\right)$

Therefore, by ind.....

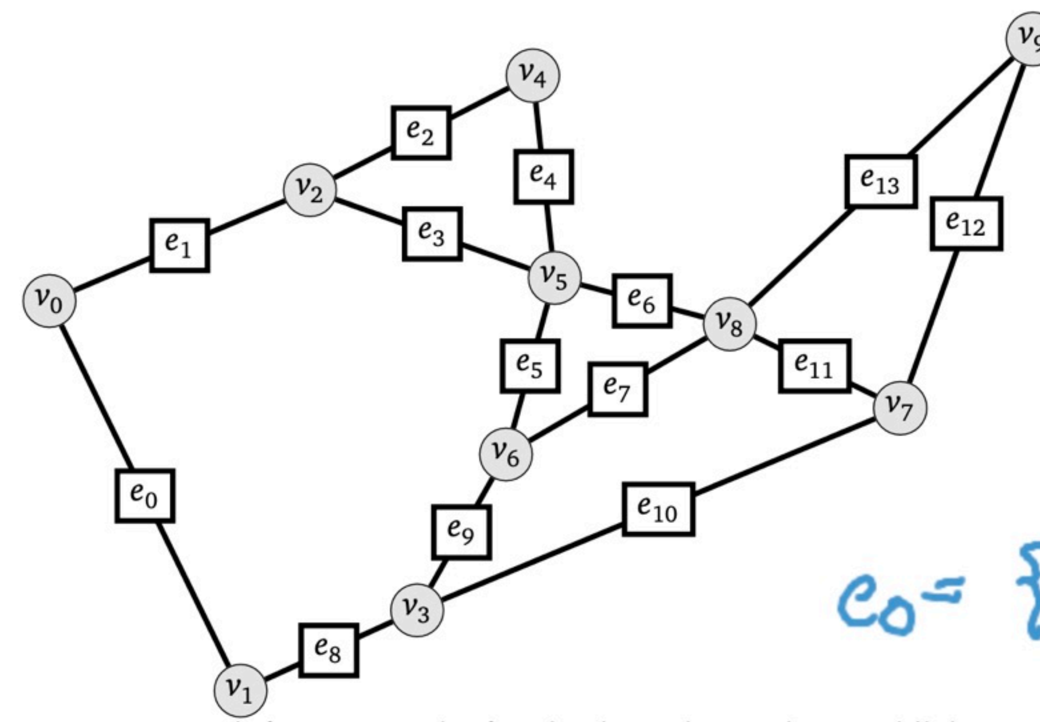
□

Goal for today:

1. Describe graphs as a pair of (1) a set of edges and (2) a set of vertices.
2. Identify properties of graphs: weighted, simple, directed.
3. Represent graphs using adjacency matrices and adjacency lists.

Our first graph!

Definition: A *graph* G is a pair of sets $G=(V,E)$ where V is a nonempty set of items called vertices (or nodes) and E is a set of 2-item subsets of V called edges.

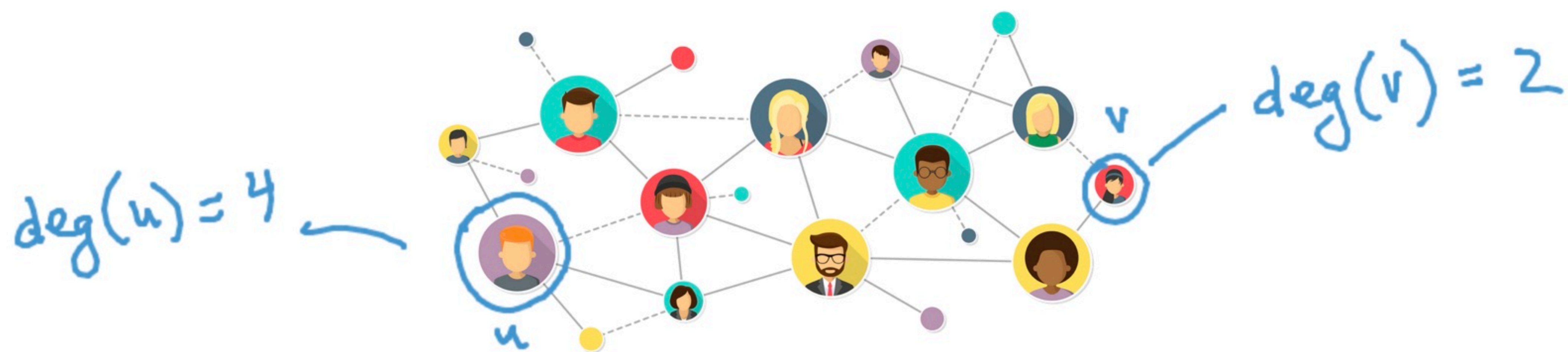


$e_0 = \{v_0, v_1\}$

might use (v_0, v_1)

- two vertices $v_i,\ v_j$ are ADJACENT if $\{v_i, v_j\} \in E$
- the edge $e=\{v_i,v_j\}$ is INCIDENT to endpoints v_i and v_j.

The *degree* of a vertex v is the number of edges incident to v, denoted $\deg(v)$.



Exercise: (two minutes) Turn and talk to a few people. Ask each other a quick question such as "what is your favorite animal or color?". Record the **number of people you talked to** m. Click a button on our website m times.

72

A	B	C	D	E	✋	❤️	👍	👎	😳	🤔	😊	🐦	🐢	🐍
3	2	5	4	4	5	6	5	7	6	3	4	6	6	7

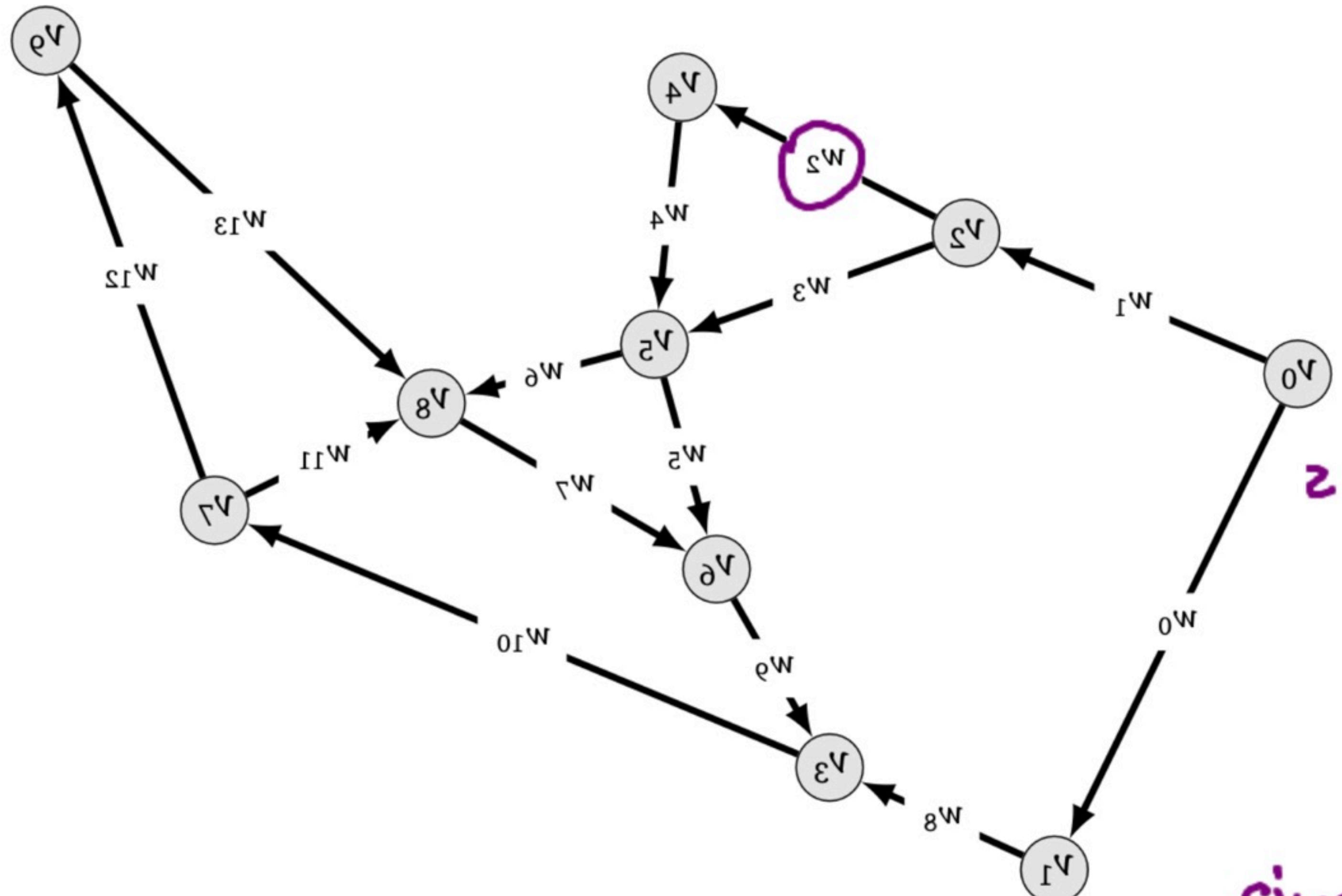
The handshaking lemma: the total sum of the vertex degrees is equal to twice the number of edges in a graph.

Proof. We use a direct proof. Let $G = (V, E)$ be a graph. Traverse every edge $e \in E$ and add the contribution of that edge $e = \{v_i,\ v_j\}$ to the total degree of the endpoint vertices v_i and v_j. The total degree can be calculated as:

$$\sum_{v \in V} \deg(v) = \text{total degree} = \sum_{e \in E} (1 + 1) = \sum_{e \in E} 2 = 2|E|.$$

$\square$

Properties of graphs: weighted, simple, directed.



loops 

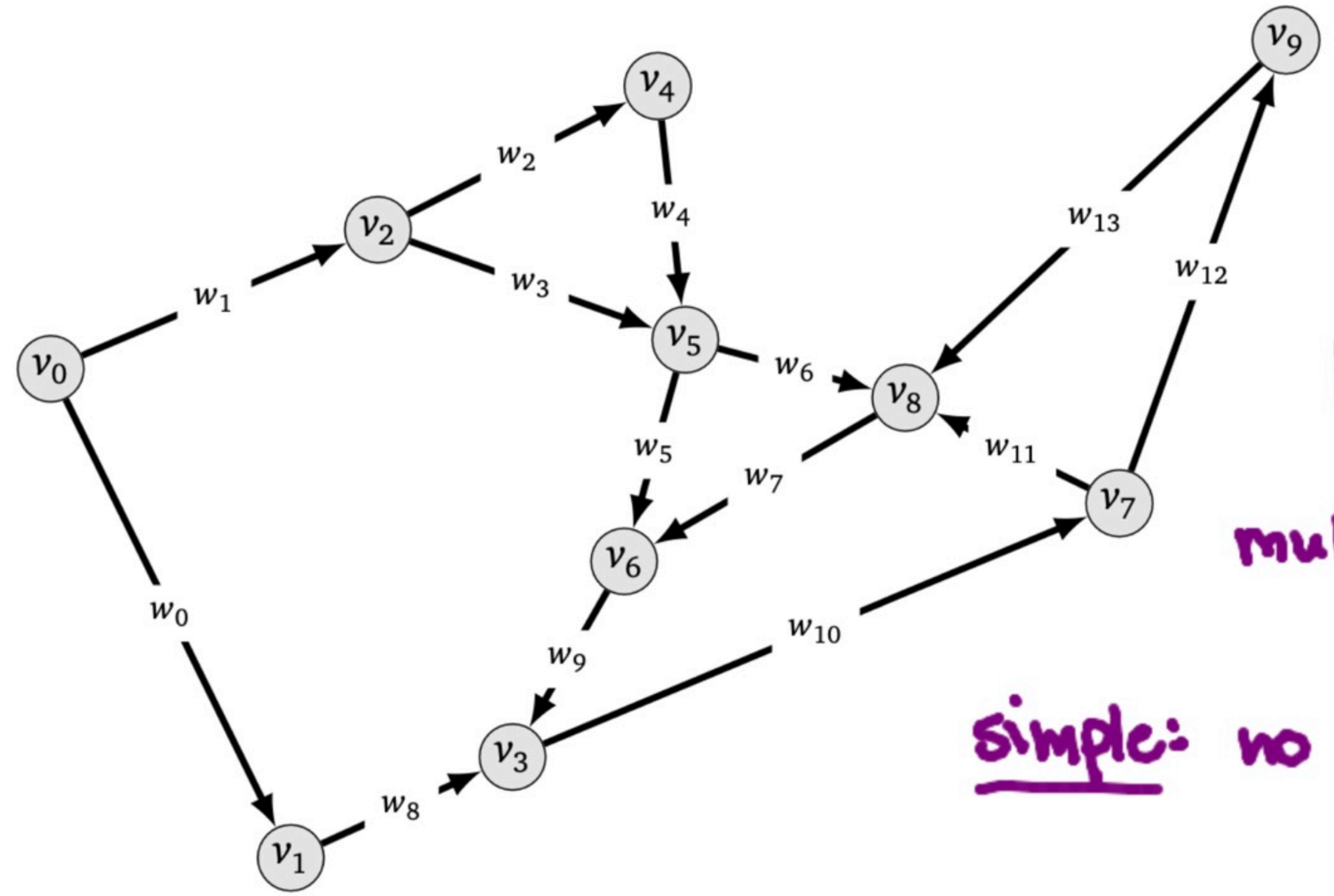
multi edges

Is this graph simple? A: Yes, B: No

simple

no loops or
multi-edges

Properties of graphs: weighted, simple, directed.



loop

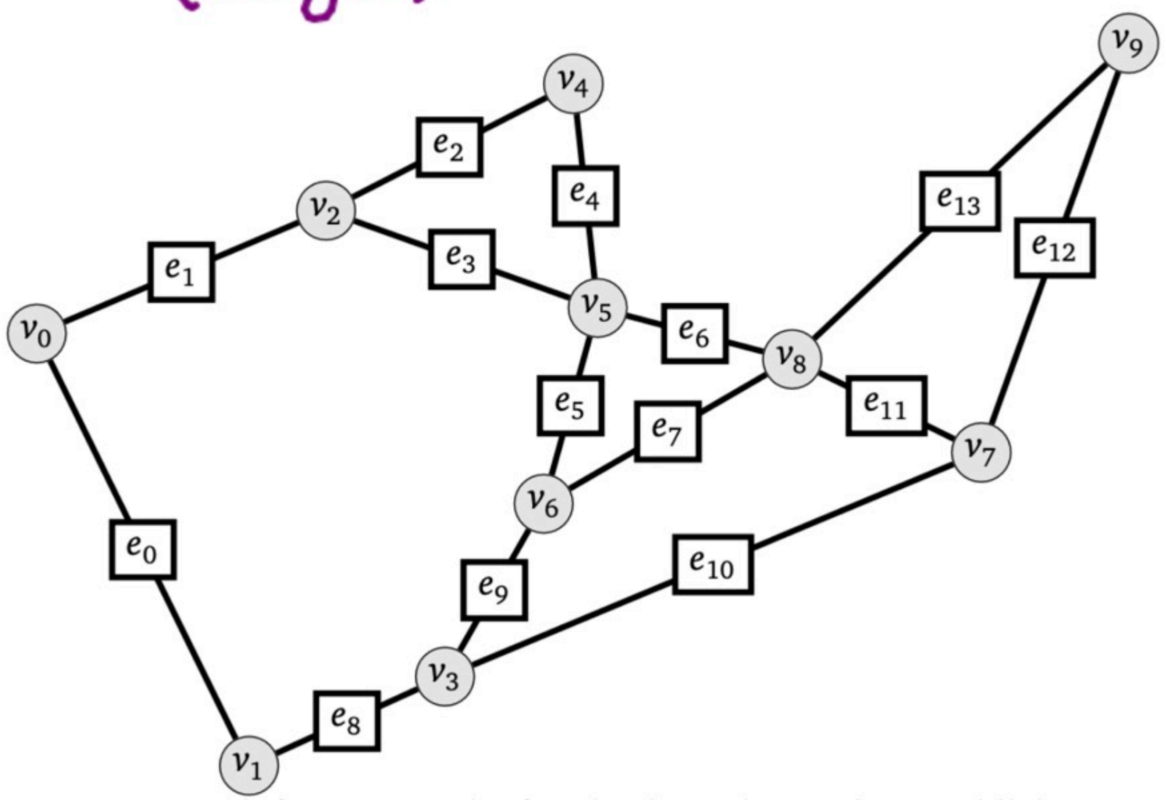
multi-edge

simple: no loops or multi-edges

Is this graph simple? A: Yes, B: No

Representing graphs with an *adjacency matrix*.

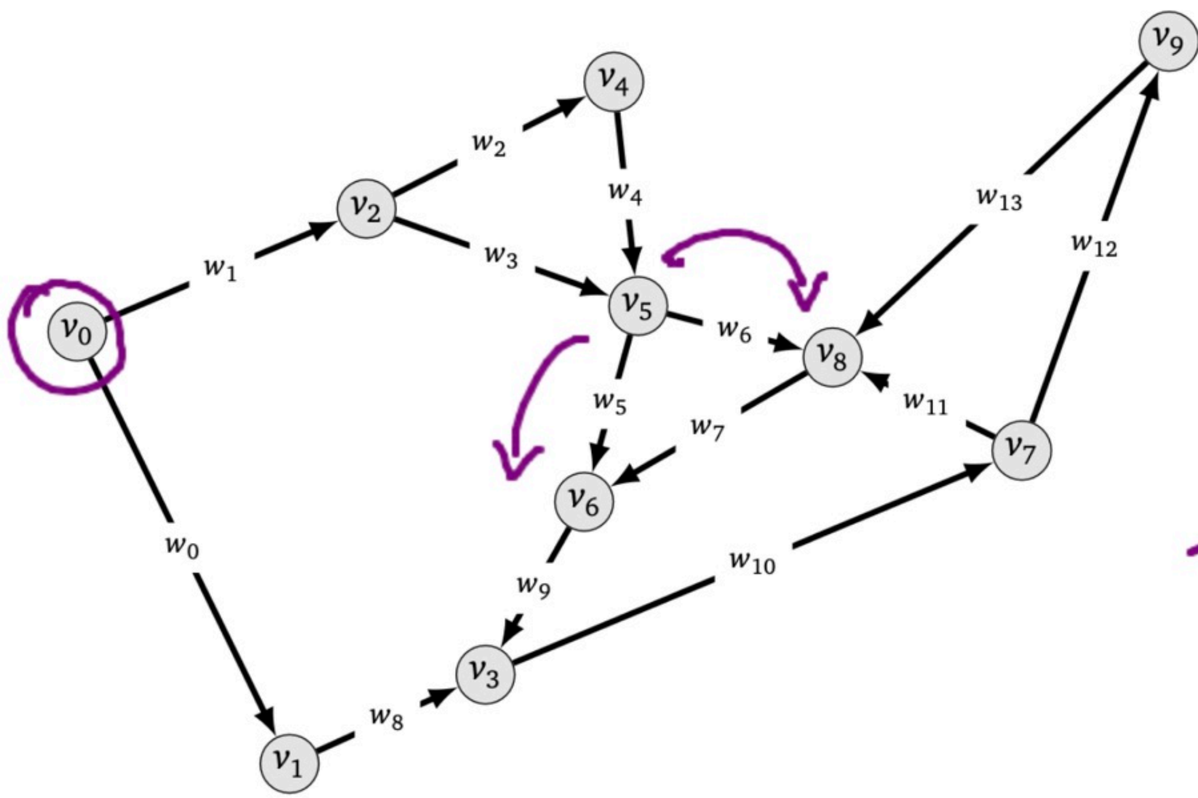
put 1 if there is an edge between vertices, 0 otherwise
(weight)



$$
\begin{array}{c}
 \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

Representing graphs with *adjacency lists*.

store list of node (or pair of (node, weight))

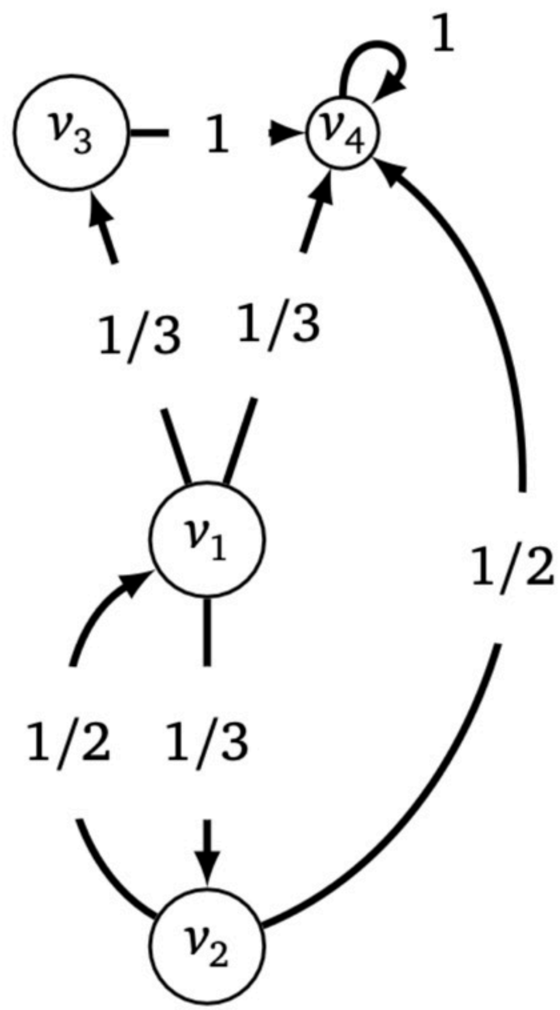


Vertex	List
v_0	$[(v_1, w_0),\ (v_2, w_1)]$
v_1	$[(v_3, w_8)]$
v_2	$[(v_4, w_2),\ (v_5, w_3)]$
v_3	$[(v_7, w_{10})]$
v_4	$[(v_5, w_4)]$
v_5	$[(v_6, w_5),\ (v_8, w_6)]$
v_6	$[(v_3, w_9)]$
v_7	$[(v_8, w_{11}),\ (v_9, w_{12})]$
v_8	$[(v_6, w_7)]$
v_9	$[(v_8, w_{13})]$

$v_0 : [(v_2, w_1), (v_1, w_0)]$

Example: write the adjacency matrix and adjacency lists for this graph.

directed
weighted



$$
\begin{array}{c|cccc}
 & v_1 & v_2 & v_3 & v_4 \\
\hline
v_1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
v_2 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
v_3 & 0 & 0 & 0 & 1 \\
v_4 & 0 & 0 & 0 & 1
\end{array}
$$

$v_1 \quad [(v_2, \frac{1}{3}), (v_3, \frac{1}{3}), (v_4, \frac{1}{3})]$

$v_2 \quad [(v_1, \frac{1}{2}), (v_4, \frac{1}{2})]$

$v_3 \quad [(v_4, 1)]$

$v_4 \quad [(v_4, 1)]$