Middlebury

CSCI 200: Math Foundations of Computing

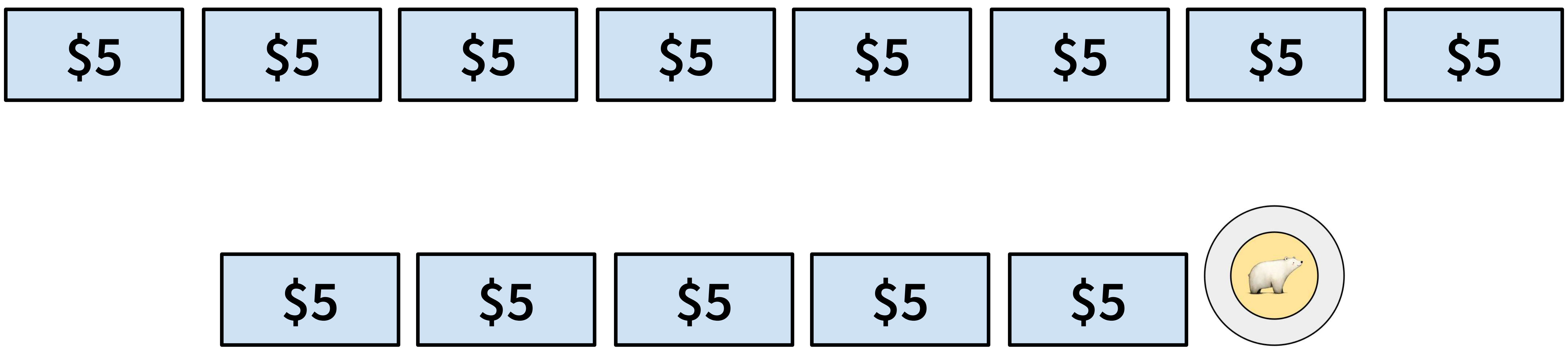
Spring 2026

Lecture 6W: Induction

Paying for stuff with only $2 coins and $5 bills.

Suppose you have unlimited toonies ($2 coins) and $5-dollar bills.

- Can you make $1?
- Can you make $3?
- Can you make $67?



Goals for today:

- State the basic steps of a proof by induction.
- Prove some simple propositions using induction.

If you have 10$ (5 x 2$), can you make 11$, 12$, 13$, 14$?

11

$5

12

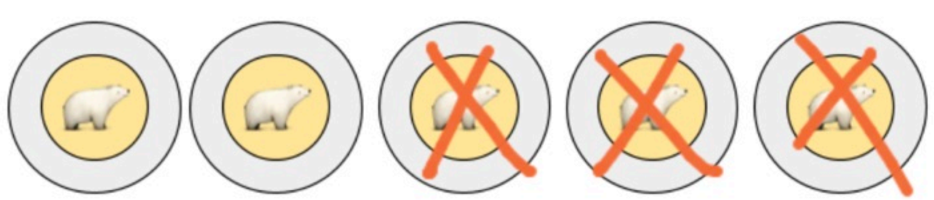
$5 $5

or + 2$

13

$5

14



$5 $5

or + $2 + $2

The Principle of Mathematical Induction

main idea: show that some predicate is true for $\overbrace{(n+1)}^{\text{some variable}}$ assuming it is true for some value n.

1. **State your method:** We use a proof by induction.

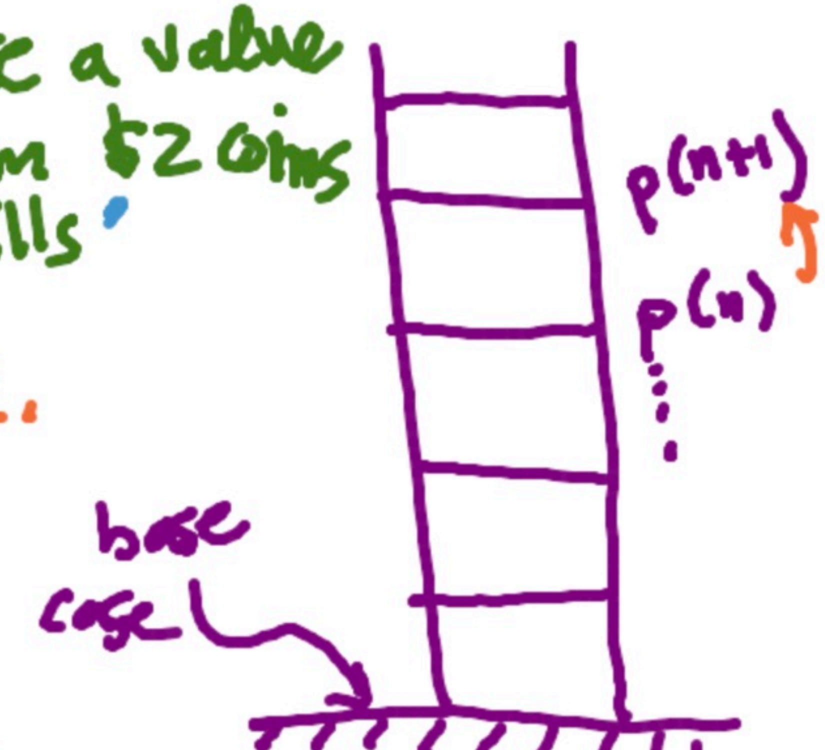
2. **Identify your induction variable:** ... on n (dollar value).

3. **State your induction hypothesis:**
   Let IH be the predicate p(n) = "we can make a value of \$n from \$2 coins and \$5 bills"

4. **Prove your base case:**
   What is the lowest value to consider.

5. **Prove the inductive case:**
   Prove p(n+1) is true, assuming p(n) is true.

6. **Conclude:**
   Therefore, by induction on n, p(n) is true for all n ≥ base case. □



An idea about 2$ coins and 5$ bills:
If $\underbrace{\text{we have} > 3\$}_{p}$, then $\underbrace{\text{we either have} \geq 2 \text{ x } 2\$ \text{ or} \geq 1 \text{ x } 5\$}_{q}$.

Proof SKETCH (practice writing out the complete proof!)

$\neg p$: less than \$3.

$\neg q$: $[< 2 \times \$2$ AND $0 \times \$5] \rightarrow$ total = \$2

$\neg q \rightarrow \neg p$ ✓

have \$2 < \$3

we'll use this as a lemma.

Proof that we can make any value $> 3\$$ using only 2\$ coins and \$5 bills.

Proof. We use a proof by induction on the dollar value n. Let the induction hypothesis be: $p(n)$ = "A value of $\$n$ can be made from \$2 coins and \$5 bills." We will prove $p(n)$ is true $\forall n > 3$.

- **Base case:** We can make $n = 4$ (the lowest dollar value) from two \$2 coins, verifying $p(4)$.
- **Inductive step:** Assume $p(n)$ is true. That is, A value of $\$n$ can be made from \$2 coins and \$5 bills. We will show that a value of n+1 can be made from \$2 coins and \$5 bills. (show p(n+1) true).

By lemma on the last slide, we always have either \$2 × 2 or \$5 × 1 (at least).

So a value of n+1 can be created by either ① subtract 2 × \$2 and add 1 × \$5 or ② subtract 1 × \$5 and add 3 × \$2.

−4   +5   −5   +6

This means that a value of \$(n+1) can be made from \$n.

Therefore, by induction on n, p(n) is true for all $n \geq 4$.

□

A numbery example: prove $3 \mid (n^3 - n),\ \forall n \geq 0.$

$n^3 - n = 3K$
$K \in \mathbb{Z}.$

Proof. We use a proof by induction on an integer n. Let the induction hypothesis be: $p(n)$ = "$3 \text{ divides } n^3 - n.$"

- **Base case:** Our base case is for $n = 0$, for which we have that 3 divides $0^3 - 0 = 0$.
- **Inductive case:** Assume $p(n)$ is true. Then 3 divides $n^3 - n$. This means $n^3 - n = 3k$ for some $k \in \mathbb{Z}$. Looking at $p(n+1)$:

$$
\begin{aligned}
(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n \cancel{+} 1 - n \cancel{-} 1 && \text{expanding} \\
&= n^3 - n + 3n^2 + 3n && \text{manipulating} \\
&= 3K + 3n^2 + 3n && \text{by } p(n) \text{ (assumption)} \\
&= 3(K + n^2 + n) && \text{factoring a } 3 \\
&= 3m && m \in \mathbb{Z}\text{; showing } p(n+1) \text{ true.}
\end{aligned}
$$

Therefore, by induction on n, $p(n)$ true for all $n \geq 0$.

$\square$

Is this proof okay?

NO

In a bag of n candies ($n \geq 1$), all candies have the same flavor.

Proof: We use a proof by induction on the induction variable n, the number of candies in a bag. Let the induction hypothesis be: $p(n) =$ *in a bag of n candies, all candies have the same flavor*.

- **Base case:** Our base case is for $n = 1$, i.e. for a single candy, it has the same flavor as itself.
- **Inductive step:** Assume that $p(n)$ is true. That is, in a bag of n candies, they all have the same flavor. Now, consider a set of $n + 1$ candies. by $p(n)$, same flavor

  $c_1 \quad c_2 \quad c_3 \cdots c_n \quad c_{n+1}$ by $p(n)$, same flavor

  Since c_1 is the same flavor as the candies: $c_2, \ \ldots, \ c_n$ and c_{n+1} has the same flavor as the candies: $c_2, \ \ldots, \ c_n$, then c_1 and c_{n+1} have the same flavor.

By induction, this means that a bag of n candies all have the same flavor. $\square$

$n+1=2$ $\quad$ $c_1 \quad c_2$ $\quad$ X nothing connects the flavors.