Middlebury

CSCI 200: Math Foundations of Computing

Spring 2026

Lecture 5W: Direct Proofs

Summary of what we can do so far:

- Use mathematical notation to represent things.
- Use a truth table to check a conclusion.
- Derive new conclusions using our 4 rules of inference.

What do we want to do today?

- Prove conclusions (propositions, theorems, lemmas).
- Divide these proofs into cases (if necessary).
- Use correct proof structure.

General structure of a proof.

can omit if direct proof and single case.

- State your plan. We use a proof by [insert proof method].
- Introduce your variables. Let n be an integer. Let $x \in \mathbb{R}$
- State your assumptions. Suppose [insert statement]. Assume [statement].
- Write your proof as an essay (not a calculation).

  use complete sentences!
- Revise and simplify.

  can you remove steps without sacrificing clarity?
- Finish.

Therefore .....

always remember this!

Method #1: Direct/Manipulate

goal: show LHS = RHS

[start by stating proof method] → optional for direct

Introduce variables + assumptions.

Start with the LHS

steps...

steps...

steps...

= RHS

don't manipulate LHS, RHS at same time.

[Example 1] Prove: *If* $\overbrace{a \mid b}^{p}$ *and* $\overbrace{b \mid c}^{q}$*, then* $\overbrace{a \mid c}^{r}$.

$p \wedge q \to r$

↑ "divides" $\frac{b}{a}$ is an integer.

proof: We use a direct proof.

Let $a, b, c \in \mathbb{Z}$.

Assume $a|b$ and $b|c$.

This means $\exists\, m, n \in \mathbb{Z}$ such that $\frac{b}{a} = m$ and $\frac{c}{b} = n$.

We want to show $a|c$. In other words $\frac{c}{a} = k$, $k \in \mathbb{Z}$.

Starting with $c = b \cdot n = (a \cdot m) \cdot n = \underbrace{(m \cdot n)}_{k} a = a \cdot k$

for some $k \in \mathbb{Z}$.

Therefore, $a|c$.

$\blacksquare$

[Example 2] Prove: *If n is an odd integer, then n^2+3n+5 is odd.*

$n=2k+1 \quad k\in\mathbb{Z}.$

proof: We use a direct proof.

Let n be an odd integer, so $n=2k+1$, $k\in\mathbb{Z}$.

We want to show n^2+3n+5 is odd. In other words, we want to show $n^2+3n+5=2m+1$ for some $m\in\mathbb{Z}$.

even+odd = odd

Starting with n^2+3n+5

$$= (2k+1)^2+3(2k+1)+5$$

$$= 4k^2+4k+1+6k+3+5$$

$$= 4k^2+10k+8+1$$

$$= 2(\underbrace{2k^2+5k+4}_{m})+1 = 2m+1$$

Therefore n^2+3n+5 is odd $\blacksquare$

Method #2: Split into cases.

$x < 0$? | $x = 0$? | $x > 0$?

method 1 | method 2 | method 3

[Example 3] **Prove:** *If $n \in \mathbb{N}$, then $1 + (-1)^n(2n-1)$ is a multiple of 4.*

<u>proof:</u> We use a proof by cases where $n \in \mathbb{N}$ is either even or odd.

<u>case 1:</u> Let n be a positive even integer.

Then $n = 2k$ for some $k \in \mathbb{Z}$.

We want to show $1 + (-1)^n(2n-1) = 4m$ for some $m \in \mathbb{Z}$.

Starting with $1 + (-1)^n(2n-1) = 1 + \underbrace{(-1)^{2k}}_{+1}(2(2k)-1)$

$$= 1 + (4k-1)$$

$$= 4k$$

Therefore $1 + (-1)^n(2n-1)$ is a multiple 4 when n is even.

<u>case 2:</u> Let n be a positive odd integer. Let $n = 2k+1$ $k \in \mathbb{Z}$.

....SKETCH....not a proof.... $1 + \underbrace{(-1)^{2k+1}}_{-1}(2(2k+1)-1) = 1-(4k+2-1)$

$$= 1-(4k+1) = -4k$$

......more [illegible]