Problem 3: inversions & indicator random variables.

Consider an ordered list containing the elements $\{1, 2, 3, \ldots, n\}$ with no repeats. An inversion is a pair (i, j) where $i < j$ but j precedes i in the list. For example, if we consider the ordered list $(3, 1, 4, 2)$ of the elements $\{1, 2, 3, 4\}$, there are 3 inversions: $(1, 3),\ (2, 3),\ (2, 4)$. If an ordering is chosen with *equal* probability from all possible orderings, we would like to determine the average number of inversions.

→ total # of inversions.

1. What is the sample space? What is the size of the sample space? What is the random variable of interest?
2. Express the random variable of interest as a weighted sum of indicator random variables.
3. Use linearity of expectation and properties of indicator random variables to calculate the average number of inversions.

3	1	4	2

1.) $n!$ sample space = all possible permutations of $\{1, \ldots, n\}$

2.) X = total # of inversions (rand. var. of interest)

X_{ij} = indicator rand. variable = $\begin{cases} 1 & \text{if there is an inversion between } (i,j) \\ 0 & \text{otherwise} \end{cases}$

$$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$$

for ind. rand variable = probability = $\frac{1}{2}$

$n-(i+1)+1 = n-i$

3.) want $E[X] = E\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_{ij}\right] \xrightarrow{\text{LOE}} \sum_{i=1}^{n}\sum_{j=i+1}^{n} E[X_{ij}] = \sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{1}{2} = \frac{1}{2}\sum_{i=1}^{n}\underbrace{\sum_{j=i+1}^{n} 1}_{n-i} = \frac{1}{2}\sum_{i=1}^{n}(n-i)$

$= \frac{1}{2}n^2 - \frac{1}{2}\overbrace{\sum_{i=1}^{n} i}^{\frac{1}{2}n(n+1)}$

$= \frac{1}{4}n(n-1)$

Problem 5: hockey stick identity

1. Prove, using induction on n, that

$$\sum_{k=m}^{n}\binom{k}{m}=\binom{n+1}{m+1}$$

for $n, m \in \mathbb{N}$ and $n \geq m$.

2. Suppose you are distributing p *identical* objects into q bins. We know, from the stars-and-bars technique, that this can be done in $\binom{p+q-1}{q-1}$ ways. Alternatively, we could distribute j objects into the first $q-1$ bins, and then distribute $p-j$ objects into the last bin. This can be done for either $j=0, j=1, \ldots$ or $j=p$ (i.e. $0 \leq j \leq p$). Use this approach of distributing j objects into $q-1$ bins ($0 \leq j \leq p$) along with the identity from part (a) to show that you arrive at the same result as the stars-and-bars approach for counting the number of ways to distribute p objects into q bins. *Hint: substitute* $n = p + m$.

1) sketch IH: $\sum_{k=m}^{n}\binom{k}{m}=\binom{n+1}{m+1}$

base: For $n=1$, $m=1$: $\sum_{k=m}^{n}\binom{k}{m}=\sum_{k=1}^{1}\binom{k}{1}=\binom{1}{1}=\frac{1!}{1!0!}=1$

show $= \binom{n+1}{m+1}=\binom{1+1}{1+1}=\binom{2}{2}=\frac{2!}{2!0!}=1$ ✓ equal

inductive: Assume $\sum_{k=m}^{n}\binom{k}{m}=\binom{n+1}{m+1}$

Show $\underbrace{\sum_{k=m}^{n+1}\binom{k}{m}}_{\text{start}}=\binom{n+1+1}{m+1}=\underbrace{\binom{n+2}{m+1}}_{\text{show this}}$

$\hookrightarrow \sum_{k=m}^{n}\binom{k}{m}+\binom{n+1}{m}=\underbrace{\binom{n+1}{m+1}+\binom{n+1}{m}}_{\text{apply def. of binomial coefficients}}$ ....algebra....

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2) stars and bars → n objects, k people $\binom{n+k-1}{k-1} \to \binom{p+q-1}{q-1}$ (here)

q-1 bins

want # ways to look like this $\sum_{k=m}^{n} \binom{k}{m}$

j objects into q-1 bins $\binom{j+q-1-1}{q-1-1} = \binom{j+q-2}{q-2}$

#ways $= \sum_{j=0}^{p} \binom{j+q-2}{q-2}$

Substitute $q-2=m$, $j+q-2=k$

for $j=0$ $k=q-2=m$

for $j=p$ $k=p+q-2=p+m$

$= \sum_{k=m}^{p+m} \binom{k}{m} = \sum_{k=m}^{n} \binom{k}{m} \underset{\text{by identity}}{=} \binom{n+1}{m+1} = \binom{p+m+1}{q-1} = \binom{p+q-1}{q-1}$

$m+1 = q-1$

Problem 6: linear recurrences.

Suppose you determined that the following function is the slowest part of a much larger codebase. You have been given the task of rewriting the result of this function into a closed-form mathematical expression to speed it up.

1. Express the result of this function as a recurrence relation.
2. What are the base cases (for $n = 0$ and $n = 1$)?
3. Determine a closed-form expression for $\texttt{function}(n)$.
4. Prove, using a proof by induction, that your function indeed returns the correct value. You may either work with the algorithm in question, or your recurrence relation from part (a).

```
function(n)

    input: n ∈ ℕ
    if n < 2    # base cases
        return 3(1 − n)
    else   # recursive case
        return 5 × function(n − 1) − 4 × function(n − 2)
```

1.) $F(n) = 5F(n-1) - 4F(n-2)$

2.) $F(0) = 3 \quad F(1) = 0$

3.) Assume $F(n) = r^n$

$r^n = 5r^{n-1} - 4r^{n-2}$

a) solve for r

b) use $F(0)$ and $F(1)$ to find c_1, c_2

Problem 7: a game with linearity of expectation.

Let's play a game! Pick a number between 1 and 6. Now roll three fair, independent dice. Calculate your earnings for this round using the following rules:

1. If your number does not come up at all, you lose a dollar.
2. If your number comes up once, you win a dollar.
3. If your number comes up twice, you win two dollars.
4. If your number comes up three times, you win four dollars.

We want to analyze your average earnings in each round of this game.

1. Let X_i be an indicator random variable that is equal to 1 if you have i matches with your chosen number after rolling the three dice. Express the earnings per round as a weighted sum of these indicator random variables.
2. What is the probability of having i matches with your chosen number?
3. Apply linearity of expectation to calculate the average earnings during each round of this game.

1.)

total # outcomes 6^3

0 matches 5^3

1 match $5^2\binom{3}{1}$

2 matches $5\binom{3}{2}$

3 matches 1

X_i is rand. variable in which you get i matches

X = amount we win per round.

$X = -1\cdot X_0 + 1X_1 + 2X_2 + 4X_3$

$E[X]$ = by LoE $= -1\,E[X_0] + 1\,E[X_1] + 2\,E[X_2] + 4\,E[X_3]$