Learning objectives:

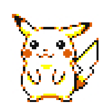
- ☐ practice some more with solving probability problems,
- ☐ calculate the probability that an event A occurs, given that an event B occurs,
- ☐ determine if two events are independent.

In the last lecture, we introduced some terms and techniques for solving probability problems. We realized that solving a probability problem can be reduced down to a counting problem, by counting the number of times an event can occur and dividing by the total number of possible outcomes. Today we will consider questions like *given that A occurs (with some probability), what is the probability that B occurs?* This is known as conditional probability - but first an example!

Example 1:
In a best 2-out-of-3 series of hockey/basketball/baseball/lacrosse/etc., suppose the probability of winning the first game is $\frac{1}{2}$. Also suppose that the probability of winning immediately after a win is $\frac{2}{3}$, and the probability of winning immediately after a loss is $\frac{1}{3}$. Ultimately, we want to answer the question: what is the probability that you win the series if (given) you win the first game? For now, just draw the probability tree and label the edges with the probabilities of winning/losing at any particular game of the series.

Solution:
The probability tree is shown in Figure 1.

Probability of winning the series?



Verify that the probability of winning the series is the sum of the winning outcomes (W,W), (W,L,W) and (L,W,W), which is $\frac{1}{3} + \frac{1}{18} + \frac{1}{9} = \frac{1}{2}$.

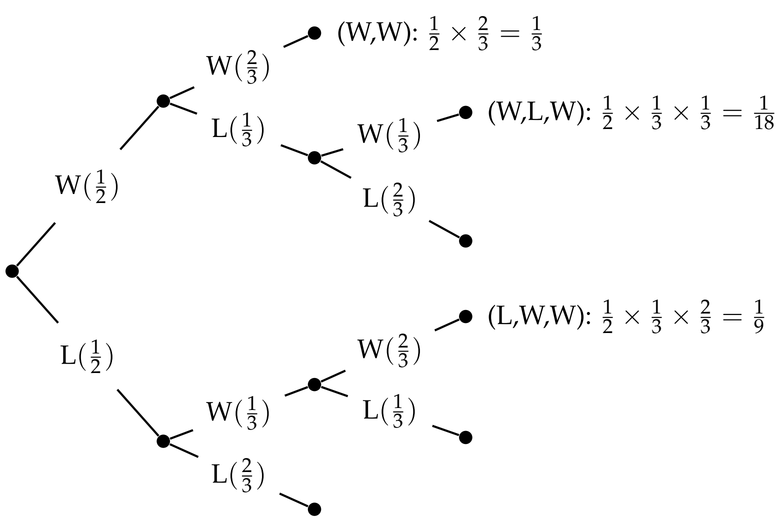


Figure 1: Probability tree for the best of 3 series.

1 Conditional probability

We are interested in things like the *probability of winning the series given that we won the first game*. It would be pessimistic to simply compute the sum of the probabilities of the (W,W) and (W,L,W) branches (which are the only branches in which we win the series after winning the first game): $\frac{1}{3} + \frac{1}{18} = \frac{7}{18}$. In fact, the sample space is reduced if we only consider the outcomes that result from winning the first game. We should normalize by the probability of winning the first game, which is $\frac{1}{2}$. This gives a probability of winning the series *given* that we won the first game as $\frac{7}{18}/\frac{1}{2} = \frac{7}{9}$. We can generalize this to calculate the probability of an event A, given that event B occurs.

What does the vertical bar | mean?

Here, the vertical bar (|) means "given." You should read $p(A \mid B)$ as "the probability of A given B".

Definition 1. *The probability of an event A occuring, given that event B occurs is*

$$p(\underbrace{A \mid B}_{A \text{ given } B}) = \frac{p(\overbrace{A \cap B}^{A \text{ and } B})}{p(B)}, \quad p(B) \neq 0. \tag{1}$$

*In other words, it is the probability of A **and** B occuring, normalized by the probability that B occurs. Since A and B are subsets of the sample space S, we represent the event that A and B occur as the event (another subset) $A \cap B$.*

In our series example, the probability of *winning the series* **and** *winning the first game* is $\frac{7}{18}$, whereas the probability of winning the first game is $\frac{1}{2}$. To determine conditional probabilities, use the same tree method as in last lecture and compute the probabilities of the events A **and** B, as well as the probability of event B. Then use Equation 1 to compute the conditional probability of A given B.

1.1 Independence

Given two events A and B, we may want to calculate the probability that both A **and** B occur, as in the numerator of Equation 1. If A and B are *independent* events, then we can calculate that probability of both A and B occurring as the product of the individual probabilities of A and B.

Definition 2. *Let S be a sample space. Let $A \subseteq S$ and $B \subseteq S$ be two events. The events A and B are independent if-and-only-if*

$$p(A \cap B) = p(A) \cdot p(B) \tag{2}$$

In order to determine whether two events are independent, we just need to determine if Equation 2 is true.

2 Applications

Let us now apply our knowledge to solve some more problems.

Example 2:
You roll two dice. What is the probability that the sum of both die adds to 10? What is the probability that the sum of both die is 10, given at least one die is a 5?

Solution:
The sample space S is all possible rolls of the two die, which we can represent as the set of all pairs where the first item in the pair is the value of the first die and the second value is that of the second di: $\{(1,1),\ (1,2),\ \ldots,(1,6),\ (2,1),\ldots,(6,6)\}$. The size of the sample space is $6 \times 6 = 36$. The event consists of all outcomes in which the sum of both dice is 10: $\{(4,6),\ (5,5),\ (6,4)\}$. The probability of getting 10 is $p(E) = \frac{|E|}{|S|} = \frac{3}{36} = \frac{1}{12}$. To calculate the probability of getting a 10 given that one die is a 5, we can first calculate the size of the sample space, which is now reduced to $\{(1,5),\ (2,5),\ (3,5),\ (4,5),\ (5,5),\ (6,5),\ (5,1),\ (5,2),\ (5,3),\ (5,4),(5,6)\}$ which has a size of 11. There is only one outcome in the event: $\{(5,5)\}$. Therefore the probability of getting 10 given that one die is 5 is $\frac{1}{11}$. Note that we could approach this by using Equation 1, by noting that the probability of getting a 5 is $p(B) = \frac{11}{36}$ and the probability of getting a 10 **and** having a 5 is $P(A \cap B) = \frac{1}{36}$, giving $P(A \mid B) = \frac{1}{11}$.

Example 3:
Assume, for this example, that families have either cats (C) or dogs (D). Let E be the set of all families with 2 pets that have 2 dogs. Let F be the set of all families with 2 pets that have at least 1 dog. Are E and F independent?

Solution:
Intuitively, these are not independent but let's verify this mathematically with Equation 2. All families with 2 pets can be represented as $S = \{\text{DD, DC, CD, CC}\}$ - the order of C and D matters, assuming the pets were adopted at different times. Families with 2 pets and two dogs can be enumerated $E = \{\text{DD}\}$. Similarly, families with 2 pets and at least 1 dog can be enumerated as $F = \{\text{DD, DC, CD}\}$. The probability of E is $p(E) = \frac{1}{4}$ whereas the probability of F is $p(F) = \frac{3}{4}$. The probability of having two dogs and at least 1 dog is $p(E \cap F) = \frac{1}{4}$. Observe that $p(E \cap F) \neq p(E) \cdot p(F) = \frac{1}{4} \cdot \frac{3}{4}$, therefore, E and F are not independent.

Example 4:
Let A be the event in which you randomly create bit strings of length 4 that begin with a 1. Let B be the event in which you randomly create bit strings of length 4 with an even number of 1's. Are A and B independent?

Solution:
To determine if A and B are independent, we should verify Equation 2. The total possible number of bit strings of length 4 (the size of the sample space) is $2^4 = 16$. The number of bit strings with length 4 that start with a 1 is 2^3 since we are free to pick any 3 bits. That is $A = \{\texttt{1000, 1001, 1010, 1100, 1011, 1100, 1110, 1111}\}$. The bit strings with length 4 that have an even number of 1's are $B = \{\texttt{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111}\}$, of which there are 8. Therefore $A \cap B = \{\texttt{1001, 1010, 1100, 1111}\}$. Then the probability of A and B is $p(A \cap B) = \frac{4}{16} = \frac{1}{4}$. Since the $p(A) = \frac{8}{16}$ and $p(B) = \frac{8}{16}$, then $p(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(A) \cdot p(B)$. So A and B are independent.