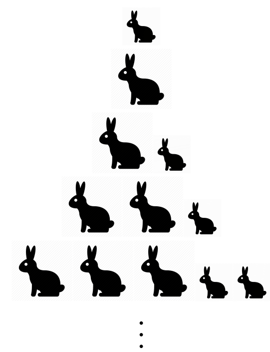
Learning objectives:

- ☐ identify linear recurrence relations,
- ☐ solve homogeneous linear recurrence relations,
- ☐ solve inhomogeneous linear recurrence relations.

In the last lecture, we introduced a few methods for solving recurrence relations: (1) guess and check and (2) expand and pray. Today, we'll restrict our attention to special kinds of recurrences that show up in the Towers of Hanoi problem and the definition of Fibonacci numbers. We'll give you some better tools for solving these kinds of recurrence relations. But first, a motivating example!



Let's say that a newly born pair of baby rabbits is placed somewhere on campus. Assume that baby rabbits mature into adult rabbits after one month and, hence, produce a new pair of baby rabbits for the next month. Also assume that adult rabbit pairs produce a new pair of baby rabbits every month (and never die). Let the small rabbits (on the right) represent a *pair* of baby rabbits, and the large ones to represent a *pair* of adult rabbits.

At the start, we have no rabbits, but add a pair of baby rabbits. After month 1, there is still only 1 pair, but they have matured into adult rabbits. After month 2, the pair of now adult rabbits breeds a new pair of baby rabbits, so there are a total of 2 pairs. After month 3, there is a new pair of baby rabbits from the adult pair and the existing baby pair has grown into adults. After month 4, the two adult pairs breed two new baby pairs, and the previous baby pair matures, giving a total of 5 pairs of rabbits.

Fibonacci numbers?

Yes! This is the same formula for computing the n^{th} Fibonacci number!

Example 1:
Let's derive a general expression for the number of rabbit pairs after n months. Observe that the total number of pairs at month n is equal to the number of pairs at month $n-2$ (since they create baby pairs at month n), plus the number of pairs in the previous generation at $n-1$ months. In other words, the total number of pairs at month n, $p(n)$, is

$$p(n) = p(n-2) + p(n-1). \tag{1}$$

This is a little more useful than just counting, but it still requires some calculation to determine the number of pairs after 12 months. Let's look for an analytic way to solve this.

Linear recurrence relations are of the form

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \cdots + a_{n-d} f(d) + g(n) \tag{2}$$

where a_1, a_2, a_{n-d} are constants and d is called the *order* of the recurrence. The function $g(n)$ is some function of n and if $g(n) = 0$, then we say that the recurrence is *homogeneous*.

1 Getting the solution to the homogeneous part

We'll start by ignoring the $g(n)$ part (and deal with it later). When looking for a solution to a linear recurrence, guess a solution of the form:

$$f(n) = r^n$$

where r is some constant, and plug it back into the recurrence (again, ignoring $g(n)$). When doing so, we obtain

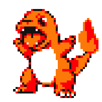
$$r^n = a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-d} r^d.$$

Bringing everything over to the left-hand-side reveals a polynomial in r:

$$r^n - a_1 r^{n-1} - a_2 r^{n-2} - \cdots - a_{n-d} r^d = 0. \qquad (3)$$

Since we want to figure what r is, we just need to solve for the roots of this equation (that's why we used r)! Equation 3 is called the *characteristic* equation. Let's go back to those rabbits.

How was I supposed to know to guess that?



We guess this form because we have a "feeling" that that the solution will grow exponentially. Don't worry, you won't have to make up any special guesses when solving linear homogeneous recurrence relations.

Always guess $\boxed{f(n) = r^n}$.

Example 2:
Is Equation 1 a homogeneous linear recurrence relation? What's the order? What are the roots of the characteristic equation?

Solution:
Our rabbit pair equation is a homogeneous second-order linear recurrence relation ($d = 2$). Substituting $p(n) = r^n$ into Equation 1 gives

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) \\ r^n &= r^{n-1} - r^{n-2} \\ 0 &= r^n - r^{n-1} - r^{n-2} \\ 0 &= r^{n-2}(r^2 - r - 1). \end{aligned}$$

If $r^{n-2} = 0$, then that means $r = 0$, which isn't very helpful, because then our solution is $p(n) = r^n = 0^n = 0$, which says the number of rabbit pairs is always zero. If $r^2 - r - 1 = 0$, then we can get a non-zero solution. Solving for r in $r^2 - r - 1 = 0$ gives the two roots of the characteristic equation:

$$r = \frac{1 \pm \sqrt{5}}{2}.$$

Great, so we can obtain the roots of the characteristic equation! Now what? Well, we need to put the solution together! But wait a second, we said the solution was $f(n) = r^n$, and we got more than one value for r!? For our rabbits, we have two solutions: $p_1(n) = (\frac{1+\sqrt{5}}{2})^n$ and $p_2(n) = (\frac{1-\sqrt{5}}{2})^n$. There's a nice result for linear recurrence relations that says if we find a solution to a linear recurrence, then any scalar multiple of that solution is also a solution. Furthermore, if we find multiple solutions to the recurrence, then any linear combination of these solutions is also a solution.

What if the degree is higher than 2?

It gets a little more difficult to solve for the roots of the characteristic equation. In this class, we will only consider first- or second-order recurrence relations. Anything that is higher-order becomes a little more difficult to solve analytically (you can always write a computer program to solve for the roots!).

Theorem 1. *If $f_1(n), f_2(n), \ldots, f_d(n)$ are solutions to the linear recurrence relation of Equation 2, then any linear combination of $f_1(n), f_2(n), \ldots, f_d(n)$ is also a solution to Equation 2. The general solution to Equation 2 can then be written as*

$$f(n) = c_1 f_1(n) + c_2 f_2(n) + \cdots + c_d f_d(n) = c_1 r_1^n + c_2 r_2^n + \cdots + c_d r_d^n.$$

So for our rabbits, we have

$$p(n) = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The last thing we need to do is figure out the constant c_1 and c_2. Luckily, we have some *initial conditions* that tell us how many pairs there are at the first few months. In general, when you need to determine c_1, c_2, c_d for a d^{th}-order recurrence relation, then you will need d initial conditions. We know that at 0 months, there are no rabbits, so $p(0) = 0$. Since they take the first month to mature (without producing baby rabbits yet), then we have 1 pair of rabbits after the first month, so $p(1) = 1$. Therefore,

$$p(0) = 0 = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = c_1 + c_2,$$
$$p(1) = 1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 = \frac{1}{2}(c_1 + c_2) + \frac{\sqrt{5}}{2}(c_1 - c_2).$$

Solving for c_1 and c_2 gives

$$c_1 = \frac{1}{\sqrt{5}}, \qquad c_2 = -\frac{1}{\sqrt{5}}.$$

So the rabbit population grows at every month as

$$p(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

How can this possibly equal an integer number of rabbits????

It's pretty amazing eh? Despite the $\sqrt{5}$'s and fractions in this equation, this *always* evaluates to an integer (for $n \in \mathbb{N}$).

2 Repeated roots

When solving for the roots of a polynomial, you may have encounted situations with repeated roots. For example when finding the roots of $r^2 - 4r + 4 = 0$, we have the root $r = 2$ twice. Our general solution would then be $f(n) = c_1 2^n$. But wait, this is a second-order recurrence relation, so we should have *two* initial conditions, and hence, two unknown constants c_1 and c_2 to solve for. Therefore, we need two unique solutions to the recurrence relation. Luckily, there's another theorem we can use when this happens.

Theorem 2. *If α is a root of the characteristic equation and is repeated m times, then α^n, $n\alpha^n$, $n^2\alpha^n$, $n^{m-1}\alpha^n$ are all solutions to the recurrence.*

Let's do an example to see where this shows up.

This looks familiar!

We actually used this same recurrence relation in the strong induction lecture.

Example 3:
Consider the following recurrence relation.

$$f(n) = 4f(n-1) - 4f(n-2)$$

with $f(0) = 1$ and $f(1) = 0$. Solve for $f(n)$ for any $n \in 0, 1, 2, \ldots$.

Solution:
We start by guessing $f(n) = r^n$ and plug this into our recurrence relation. We get

$$r^n - 4r^{n-1} + 4r^{n-2} = (r-2)^2 = 0,$$

which has the root $r = 2$ twice. By Theorem 2, the two solutions to the recurrence are

$$f_1(n) = 2^n, \qquad f_2(n) = n2^n.$$

and the general solution to the recurrence is

$$f(n) = c_1 2^n + c_2 n 2^n.$$

Applying the initial conditions $f(0) = 1$ and $f(1) = 0$ gives the two equations we need to determine c_1 and c_2:

$$f(0) = 1 = c_1$$
$$f(1) = 0 = 2c_1 + 2c_2.$$

Therefore, $c_1 = 1$ and $c_2 = -1$. The solution to the recurrence is then

$$f(n) = (1-n)\,2^n,$$

which can be verified using a proof by strong induction.

3 Solving for the inhomogeneous part (optional)

What if I kept adding a new pair of baby rabbits every month? Then the recurrence relation would no longer be homogeneous because we would need to account for the extra 1 pair every month. The recurrence relation of Equation 1 then becomes

$$p(n) = p(n-2) + p(n-1) + 1$$

which means $g(n) = 1$. When you have an inhomogeneous recurrence relation, you first need to solve the inhomogeneous part, and then determine the *particular solution* $f_p(n)$, which accounts for the inhomogeneous part.

$g(n)$	guess for $f_p(n)$
c	k
$c_1 n + c_2$	$k_1 n + k_2$
$c_1 n^2 + c_2 n + c_3$	$k_1 n^2 + k_2 n + k_3$
cr^n	kr^n

Table 1: Guesses for the particular solution $f_p(n)$ given a form for the inhomogeneous part $g(n)$.

Example 4:
Find the solution to the recurrence relation $p(n) = p(n-2) + p(n-1) + 1$ with $p(0) = 0$ and $p(1) = 1$.

Solution:
We can start by recycling the solution to the homogeneous part, since this doesn't change. In order to solve for the particular solution $f_p(n)$, we will *guess* a solution of the form $f_p(n) = k$ where k is some constant. We guess this because 1 is a constant, so the $f_p(n)$ should have a similar form. We then substitute $f_p(n)$ into the recurrence relation, giving

$$f_p(n) = f_p(n-1) + f_p(n-2) \rightarrow k = k + k + 1$$

Therefore $k = -1$. The solution to the recurrence is then

$$f(n) = f_h(n) + f_p(n) = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n - 1.$$

Using the initial conditions $p(0) = 0$ and $p(1) = 1$ gives the constants $c_1 = (3+\sqrt{5})/(2\sqrt{5})$ and $c_2 = (\sqrt{5}-3)/(2\sqrt{5})$.

In the last example, we guess that $g(n)$ was some constant. In general, you should assume a form for $f_p(n)$ that looks like $g(n)$. Table 1 summarizes a few common guesses for a particular solution, given a form for $g(n)$. If a guess fails (after you try to match coefficients), then guess substitute your polynomial for one that is one degree higher, as in the next example.

Example 5:
Find the general solution to the recurrence relation $f(n) = 2f(n-1) + 2^n$ with $f(0) = 1$.

Solution:
This is a first-order linear recurrence relation. Starting by guessing $f(n) = r^n$ gives the characteristic equation for the homogeneous part: $r - 2 = 0$. We have only one root: $r = 2$. The solution to the homogeneous part is then $f_h(n) = 2^n$. The particular solution can be obtained by guessing $f_p(n) = k2^n$. Substituting this into the recurrence relation gives $k2^n = 2k \cdot 2^{n-1} + 2^n$. This means that $k = k + 1$ which means our guess failed. As per the recommendation, we should then guess a polynomial (instead of k, which is constant) that is linear: $f_p(n) = (k_1 n + k_2)2^n$. Doing so gives

$$(k_1 n + k_2)2^n = 2(k_1(n-1) + k_2)2^{n-1} + 2^n = (k_1(n-1) + k_2)2^n + 2^n.$$

Since $2^n \neq 0$, then we have $k_1 n + k_2 = k_1(n-1) + k_2 + 1$. Matching coefficients for the constant terms gives $k_2 = k_2 - k_1 + 1$, which means $k_1 = 1$, and we can pick $k_2 = 0$. The general solution is then

$$f(n) = c_1 2^n + n2^n.$$

Applying the initial condition $f(0) = 1$ yields $c_1 = 1$, so our solution to the recurrence is finally

$$f(n) = 2^n + n2^n.$$

Check your work!

If you have time (you don't have to), it's a good idea to check your solution indeed solves the recurrence relation using a proof by induction (or strong induction if necessary).