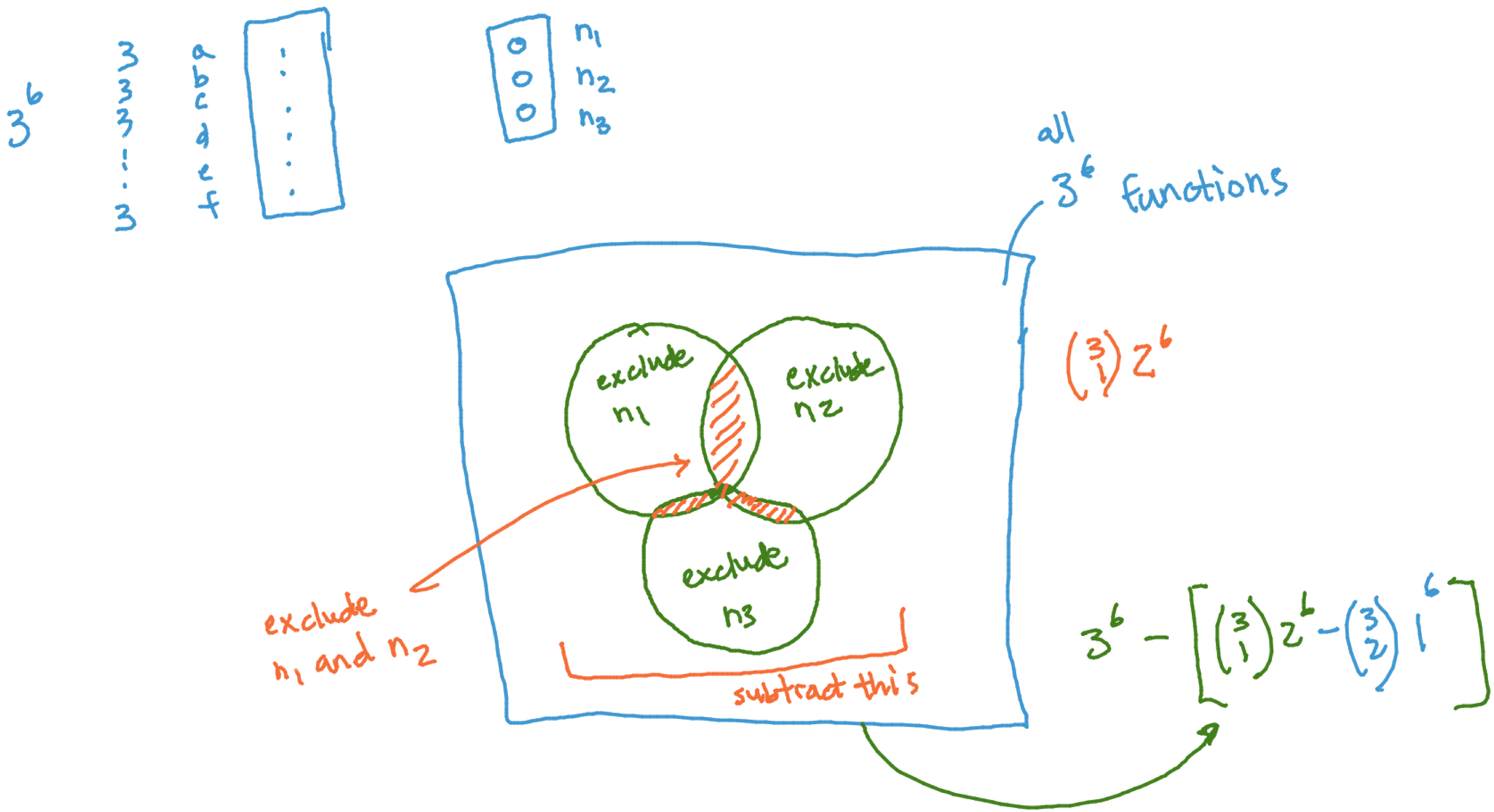
Why do we add 3 when counting surjective functions in Problem 3 of Midterm 2?



Goals for today:

- Solve homogeneous linear recurrence relations.
- Practice converting a problem description to a recurrence relation.

```python
def fib(n: int) -> int:
    if n <= 1:
        return n
    return fib(n - 1) + fib(n - 2)
```

initial conditions

two ingredients: 1) base case(s)

2) recursive step

$$f(n) = f(n-1) + f(n-2)$$

$$f(0) = 0$$

$$f(1) = 1$$

Determining a closed-form expression for Fibonacci numbers.

$f(n) = f(n-1) + f(n-2)$ with $f(0) = 0$ $f(1) = 1$

<u>guess:</u> $f(n) = r^n$

plug into recurrence. (need to find r)

$r^n = r^{n-1} + r^{n-2}$

$r^n - r^{n-1} - r^{n-2} = 0$ factor r^{n-2}

$r^{n-2}\underbrace{\left[r^2 - r - 1\right]}_{=0} = 0$ characteristic equation of degree/order 2

$r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{1-(-4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$ $\rightarrow$ two functions

$\rightarrow r_1$

$f_1(n) = \left[\frac{1+\sqrt{5}}{2}\right]^n$

$f_2(n) = \left[\frac{1-\sqrt{5}}{2}\right]^n$

<u>thm:</u> linear combination of $f_1(n)$ and $f_2(n)$ is also a solution.

$f(n) = c_1\left[\frac{1+\sqrt{5}}{2}\right]^n + c_2\left[\frac{1-\sqrt{5}}{2}\right]^n$

c_1, c_2 constants.

Determining a closed-form expression for Fibonacci numbers.

$$f(n) = c_1\left[\frac{1+\sqrt{5}}{2}\right]^n + c_2\left[\frac{1-\sqrt{5}}{2}\right]^n$$

how do we get c_1, c_2?

base cases

$f(0) = 0$

$f(1) = 1$

$\underline{n=0:}$ $\quad 0 = c_1 + c_2 \rightarrow c_2 = -c_1$ plug into find c_1,

$\underline{n=1:}$ $\quad 1 = c_1\left[\frac{1+\sqrt{5}}{2}\right] + c_2\left[\frac{1-\sqrt{5}}{2}\right]$

$$c_1 = \frac{1}{\sqrt{5}}$$

$$c_2 = \frac{-1}{\sqrt{5}}$$

$\underline{\text{final solution:}}$ $\quad f(n) = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$

What if we have this recurrence relation?

$F(n) = r^n$

$$f(n) = 4f(n-1) - 4f(n-2), \quad \text{with} \quad f(0) = 1, \quad \text{and} \quad f(1) = 0$$

$r^n = 4r^{n-1} - 4r^{n-2}$

$r^2 - 4r + 4 = 0 \qquad (r-2)^2 = 0$

thm: with repeated roots

$n \cdot f_1(n)$ is also a solution.

$r = 2$

What are the roots of the characteristic equation for this recurrence? (slido.com #3776496)

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- r = 2, r = -2
- r = 2, r = 4
- r = 2, r = 2
- r = 2, r = 1

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$f(n) = c_1 2^n + c_2 \cdot n \cdot 2^n$

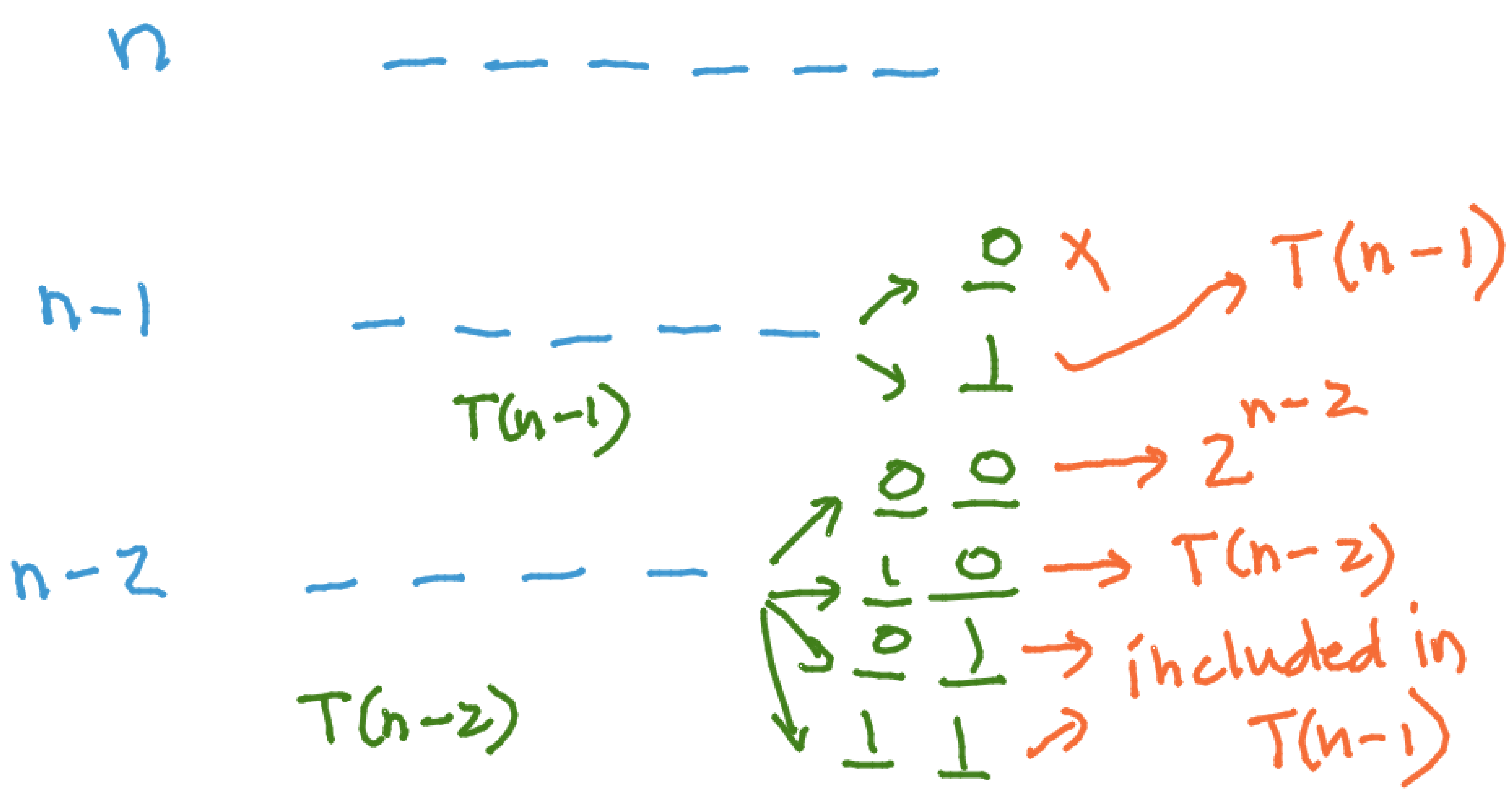
using $f(0) = 1$, $f(1) = 0$

$\boxed{f(n) = (1-n)2^n}$

Exercise: more bit strings!

Let $T(n)$ be the number of bit strings of length n that have two consecutive zeros.
Consider a recurrence relation for $T(n)$.

1. What is the base case(s) for the recurrence relation? $n=1$? $n=2$? $T(1)=0$ $T(2)=1$
2. What are the recursive conditions for the recurrence relation? $T(n)$ in terms of n, $T(n-1)$, $T(n-2)$
3. Use the recurrence relation to calculate $T(5)$.



$$\boxed{T(n) = T(n-1) + T(n-2) + 2^{n-2}}$$

$T(5) = 19$