Why do we add 3 when counting surjective functions in Problem 3 of Midterm 2?

\[ 3^6 - \left( \binom{3}{1} \cdot 2^6 - \binom{3}{2} \cdot 1^6 \right) \]
Goals for today:

- Solve homogeneous linear recurrence relations.
- Practice converting a problem description to a recurrence relation.

```python
def fib(n: int) -> int:
    if n <= 1:
        return n
    return fib(n - 1) + fib(n - 2)
```

Two ingredients:
1) base case(s)
2) recursive step

\[ F(n) = F(n-1) + F(n-2) \]

- \[ F(0) = 0 \]
- \[ F(1) = 1 \]
Determining a closed-form expression for Fibonacci numbers.

\[ f(n) = f(n-1) + f(n-2) \quad \text{with} \quad f(0) = 0 \quad f(1) = 1 \]

\[ r^n = r^{n-1} + r^{n-2} \]

\[ r^n - r^{n-1} - r^{n-2} = 0 \quad \text{factor} \quad r^{n-2} \]

\[ r^{n-2} \left[ r^2 - r - 1 \right] = 0 \]

\[ r^2 - r - 1 = 0 \]

r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{5}}{2}

\[ f(n) = c_1 \left[ \frac{1 + \sqrt{5}}{2} \right]^n + c_2 \left[ \frac{1 - \sqrt{5}}{2} \right]^n \]

\( c_1, c_2 \) constants.
Determining a closed-form expression for Fibonacci numbers.

\[ f(n) = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

**how do we get \( c_1, c_2 \)?**

**base cases**

\[ f(0) = 0 \]
\[ f(1) = 1 \]

\[ \frac{n=0}{n=1}: \quad 0 = c_1 + c_2 \quad \Rightarrow \quad c_2 = -c_1 \quad \text{plug into find} \quad c_1. \]

\[ 1 = c_1 \left( \frac{1 + \sqrt{5}}{2} \right) + c_2 \left( \frac{1 - \sqrt{5}}{2} \right) \]

\[ c_1 = \frac{1}{\sqrt{5}} \]

\[ c_2 = -\frac{1}{\sqrt{5}} \]

**final solution:** \[ f(n) = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \]
What if we have this recurrence relation?

\[ f(n) = 4f(n-1) - 4f(n-2), \text{ with } f(0) = 1, \text{ and } f(1) = 0 \]

\[ r^n = 4r^{n-1} - 4r^{n-2} \]

\[ r^2 - 4r + 4 = 0 \quad (r-2)^2 = 0 \]

What are the roots of the characteristic equation for this recurrence?

(slido.com #3776496)

- \( r = 2, r = -2 \)
- \( r = 2, r = 4 \)
- \( r = 2, r = 2 \)
- \( r = 2, r = 1 \)

Voting as Anonymous

\[ f(n) = c_1 2^n + c_2 n 2^n \]

\[ f(n) = (1-n) 2^n \]

\[ \text{thm: with repeated roots} \quad r = 2 \]

\[ f(n) = c_1 2^n + c_2 n 2^n \]

\[ f(n) = (1-n) 2^n \]
Exercise: more bit strings!

Let $T(n)$ be the number of bit strings of length $n$ that have two consecutive zeros. Consider a recurrence relation for $T(n)$.

1. What is the base case(s) for the recurrence relation?
2. What are the recursive conditions for the recurrence relation?
3. Use the recurrence relation to calculate $T(5)$.

\[ T(n) = T(n-1) + T(n-2) + 2^{n-2} \]

$n = 1$? $n = 2$? $T(1) = 0$ $T(2) = 1$

$T(n)$ in terms of $n$, $T(n-1)$, $T(n-2)$

$T(5) = 19$