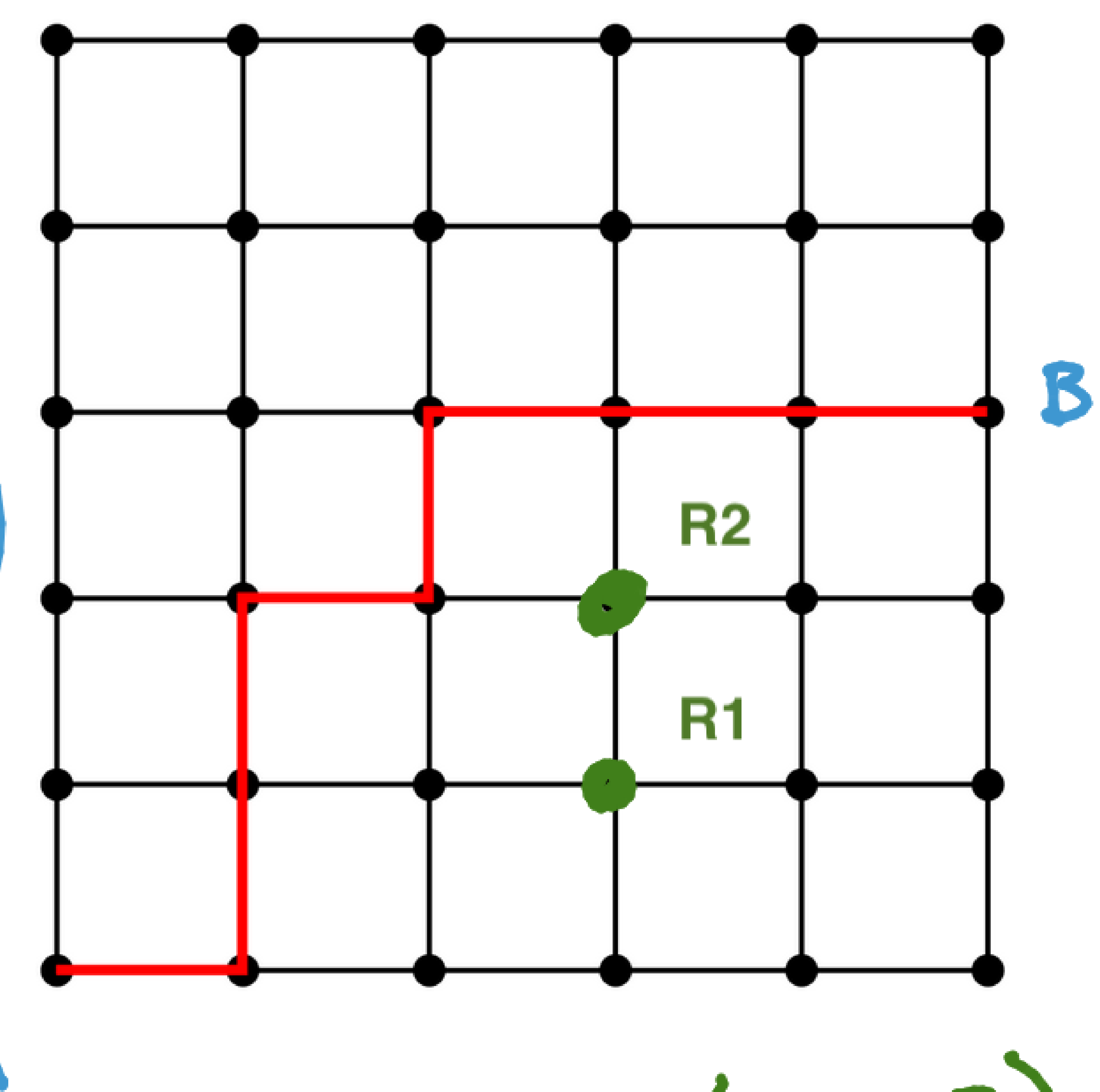
A little more about lattice paths!

$$\frac{8!}{5!\,3!} = \frac{8!}{3!\,5!}$$

paths $A \to B$: $\binom{8}{5} = \binom{8}{3}$



subtract # paths through R_1: $(A \to R_1) \times (R_1 \to B)$

subtract # paths through R_2: $(A \to R_2) \times (R_2 \to B)$

$\to$ need to add back in paths through both: $(A \to R_1) \times (R_1 \to R_2) \times (R_2 \to B)$

Problem 1: tournament graphs

- (a) How many directed graphs are there with n vertices? You may assume that self-loops and multi-edges are allowed.

n=1 n=2 n=3

1 4 9

\# possible edges: n^2

↳ each edge is either included or excluded from a graph → 2 options

$\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{n^2} = \boxed{2^{n^2}}$

- (b) How many tournament graphs are there with n vertices? A tournament graph $G = (V,\ E)$ has no self-loops and contains edges that are either directed from $u \to v$ or $v \to u$, $\forall u, v \in V$.

\# possible edges: n vertices $\binom{n}{2}$

"in how many ways can I pick vertices for an edge?"

2 options for direction of edge $u \to v$ or $v \to u$

$\boxed{2^{\binom{n}{2}}}$

Problem 2: more cards

$$\frac{48!}{2!\,46!} = \frac{48\cdot 47}{2}$$

- (a) Find the number of 5-card hands (from a deck of 52) with *exactly* three aces.

  Examples: $\{A\diamondsuit, A\spadesuit, A\clubsuit, 3\heartsuit, J\clubsuit\}$ but not $\{A\diamondsuit, A\spadesuit, A\clubsuit, 3\heartsuit, A\heartsuit\}$.

  $\underline{A_1}\ \underline{X}\ \underline{A_2}\ \underline{A_3}\ \underline{y}$

  total $= \boxed{4\binom{48}{2}}$ $= 4512$

  1) how many ways to pick A suit? (ace) $\binom{4}{3} = \frac{4!}{3!\,1!} = 4$

  2) how many cards left to pick? $52 - \underset{\text{(aces)}}{4} = 48 \to$ pick 2 $\binom{48}{2}$

- (b) How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

  $\underset{P_1}{\text{9H}}\ \underline{x}\ \underset{P_2}{\text{9S}}\ \underline{y}\ \underline{z}$

  1) # possible value of pair: 13

  2) # possible suits for pair: $\binom{4}{2}$

  3) possible values for x, y, z: $\binom{12}{3}$

  4) possible suit for x, y, z: $4\times4\times4$

  $$\boxed{\text{total} = 13\binom{4}{2}\binom{12}{3}4^3}$$

Problem 3: more bit strings

How many bit strings of length 10 have (i) exactly three 0s? (ii) at least seven 1s? (iii) exactly three 0s or start with a 1?

total # bit strings of length 10: 2^{10}

(i) _ 0 _ _ _ 0 0 _ _ _ _

$\binom{10}{3}$

(ii) 1 _ 1 1 1 1 _ 1 1 _

password problem from last class (6-8 characters)

$P_n \longrightarrow P_6 + P_7 + P_8$

$$\boxed{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} = 176}$$

bit strings with exactly 7 ones.

(iii) 1 _ _ 0 _ 0 0 _ _ _

↳ 2^9 bit strings that start with 1

$\binom{10}{3} + 2^9 -$ (# bit strings with exactly 3 zeros and start with 1)

$$\boxed{\binom{10}{3} + 2^9 - \binom{9}{3}}$$