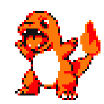
**Learning objectives:**

- ☐ Count permutations and combinations of subsets.
- ☐ Practice, practice practice!

Last lecture, we focused on counting sets using the product and addition rules, as well as the principle of inclusion-exclusion. Today, we're going to continue counting, but now discuss the various ways we can count *subsets*. We'll look at ways to select elements from some larger set to form subsets, and subsequently count the number of these subsets, as in the example below.

**So many options!**



I know. . . Sometimes I would prefer to have a smaller menu with at most 20 to 30 options.


**Example 1:**
A popular approach to dining these days consists of (1) pick a **base**, (2) pick a **protein**, (3) pick a **sauce**, (4) add some **toppings**. For example, CAVA boasts a large number of possible combinations.

When selecting a base, you can pick one of 6 options and then 1 of 7 protein sources. After picking one of 6 sauces, you can then pick 3 toppings from a set of 13. How many possible meals are there? Assume you do not double-up on toppings – for example, you don't double- or triple-up on sunflower seeds.

**Solution:**
The hardest question to answer is: *in how many ways can we pick* 3 *toppings from a set of* 13 *options*? Since we are not doubling- or tripling-up on toppings, we need to be a bit careful when counting in how many ways we can select toppings. Suppose we pick a first topping from the set of 13. When picking the second, we now have 12 choices. Similarly, when picking the third, we would have 11 choices. So overall, we have $13 \times 12 \times 11$ ways to pick 3 toppings. But wait a second. If we pick (1) seeds, (2) olives and (3) peppers, then this is the same as (1) olives, (2) peppers and (3) seeds. So we would overcount the possible combinations of toppings. What we need to do is divide by the number of times we can permute any particular selection of 3 toppings. In how many ways can I arrange 3 toppings? This is similar to the way we selected toppings in the first place: pick one of three, then there are two remaining, and then finally one. In other words $3 \times 2 \times 1$ ways we can arrange these three toppings. The total number of ways we can pick 3 unique toppings from a list of 13 is then

$$\text{\# topping arrangements} = \frac{\text{\# ways to select 3 toppings from 13}}{\text{\# ways to arrange these 3 toppings}} = \frac{13 \times 12 \times 11}{3 \times 2 \times 1} = 286.$$

The total possible number of meals is then $6 \times 7 \times 6 \times 286 = 72,072$ (CAVA probably uses a different way to pick ingredients).

## 1 Permutations

In the last example, we introduced the concept of a *permutation*.

**Definition 1.** *A **permutation** is an ordered arrangement of distinct objects. Given $n$ objects, there are $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$ ways to permute these objects.*

Counting permutations effectively reduces to using the product rule (from last lecture), in which our number of options decreases (by one) every time we pick an object. This also raises the question: how can we pick $k$ objects from a set of $n$ objects? For example, in how many ways can I assign 5 players to 5 positions if the team consists of 12 players?

**Definition 2.** *A $k$-**permutation** is an ordered arrangement of $k$ objects out of $n$.*

We actually used $k$-permutations (with $k = 3$) when picking toppings. When assigning player positions, after picking a player for the first position (of which there are 12 choices), I then have 11 choices for the second player, 10 choices for the third, 9 for the fourth, and 8 for the fifth player. This gives a total of $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$ ways, which is equal to $\frac{12!}{7!}$. We can formalize this into the following theorem.

**Theorem 1.** *The number of $k$ permutations of a set of $n$ distinct objects is*

$$P(n,k) = \frac{n!}{(n-k)!} = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1), \quad 0 \le k \le n. \tag{1}$$

Let's now practice with some examples.

**Example 2:**
In a contest with 100 contestants, in how many ways can we award $1^{\text{st}}$, $2^{\text{nd}}$ and $3^{\text{rd}}$ place?

**Solution:**
We need to pick 3 unique contestants from a set of 100, so $P(100,3) = 100!/97! = 970,200$.

**Example 3:**
How many permutations of $A$, $B$, $C$, $D$, $E$, $F$, $G$ have letters $A$ and $B$ next to each other?

**Solution:**
Let's treat $AB$ and $BA$ as blocks since these are the only two ways $A$ and $B$ can be next to each other. This means we have 6 blocks to arrange, which gives $6! = 720$ arrangements. However, we need to account for either using the $AB$ or $BA$ blocks, meaning there are a total of $720 + 720 = 1440$ arrangements.

## 2 Combinations

When selecting toppings at the beginning of the lecture, we noted that we didn't want to overcount the number of arrangements. Specifically, we were looking for *combinations* of toppings, given $n$ possibilities.

**Definition 3.** *A **combination** is an unordered subset of a set. A k-**combination** is an unordered selection of k distinct elements from a set, i.e. a subset of size k.*

We count count combinations with the following theorem.

**Theorem 2.** *The number of k-combinations of a set with $n > 0$ elements $0 \leq k \leq n$ is*

$$C(n,k) = \frac{n!}{k!(n-k)!} \tag{2}$$

Because this comes up so often, it has a special notation: $C(n,k) = \binom{n}{k}$ which is called the **binomial coefficient** and read as "$n$ choose $k$". Recall the example in which we assigned players to positions. In that example, it mattered which player we assigned to every position. But what if we just want to explore the possible number of starting line-ups, ignoring positions? Since we don't want to overcount a specific selection of players, then we need to divide by 5!, which gives $C(12,5)$ or $\binom{12}{5}$.

$n$ **choose** $k$

Remember this as *choosing* $k$ elements from a set of $n$ elements.

**Corollary 1.** *The number of combinations of choosing k items from n items is equal to the number of combinations of $n-k$ items from n.*

*Proof.* We need to verify that $C(n,k) = C(n,n-k)$. We can do so by plugging $n$ and $k$ into Equation 2, which gives

$$C(n,k) = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = C(n,n-k).$$

□

**Example 4:**
Given an $5 \times 5$ grid, suppose we start at the origin $(0,0)$ and want to get to the point $(5,3)$. In how many different ways can we go from the origin $(0,0)$ to the point $(5,3)$, assuming that we can only move in the $+x$ or $+y$ directions at each step?

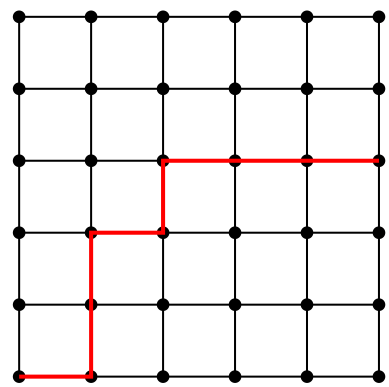
**Solution:**
To reach $(5,3)$ from the origin $(0,0)$, we need a total of 8 moves: 5 in the $x$-direction and 3 in the $y$-direction. We have two options for each move: either in the $+x$ direction or the $+y$ direction. Ultimately, we need to create sequences of length 8:

- In how many ways can we choose 5 of those 8 steps to be in the $+x$ direction? $C(8,5)$.
- In how many ways can we choose 3 of those 8 steps to be in the $+y$ direction? $C(8,3)$.

To get the total number of paths, you can either count the number of ways to go 5 steps in the $+x$ direction or the ways to go 3 steps in the $+y$ direction. Both give the same number of paths since $C(8,5) = C(8,3)$. There is a total number of 56 possible paths.

**What if there is a boulder?**

Great question! If there is a boulder at the point $(3,2)$, then we cannot pass through that point on our way to $(5,3)$. We would then need to subtract the number of paths that go through that point. The number of paths that go from $(0,0)$ to $(3,2)$ is $C(5,3) = 10$. The number of paths that go from $(3,2)$ to $(5,3)$ is $C(3,2) = 3$. The total number of paths that go *precisely* from $(0,0)$ to $(5,3)$, through the boulder is $C(5,3) \cdot C(3,2)$ by the product rule. Therefore, there are $C(8,3) - C(5,3) \cdot C(3,2) = 56 - 30 = 26$ paths we can take to reach the end point.



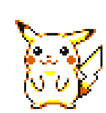
**Example 5:**
Pascal's triangle is an important application of finding combinations. It determines the coefficients which arise from binomial expansions. Specifically, if we want to calculate $(x+y)^n$, we could either expand it, or apply the binomial theorem to determine the coefficients of a particular term in the resulting polynomial. The binomial theorem states that

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Determine the coefficient of $x^5 y^{20}$ in $(x+y)^{25}$.

**Solution:**
Note that $n = 25$ and we are interested in the case when $j = 5$, so $C(25,5) = 53,130$.

**Practice, practice, practice!**



Coming up with a solution to a counting problem can be difficult and sometimes frustrating if you take the wrong approach. The best thing to do is practice!