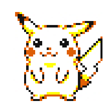
Learning objectives:

- ☐ Classify a given function as either surjective, injective or bijective,
- ☐ Count infinite sets.

We're going to start talking about functions today, which will lead into our discussion about counting. Before doing so, think about the following problem, which was famously proved in 1936 by Alan Turing.

Terminating programs



For example, the program

```
print "Hello world!"
```

terminates, but the program

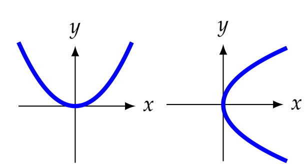
```
while (True) print "Hello world!"
```

does not terminate.

Example 1:
Given an arbitrary computer program p with an input x: is it possible to determine whether p will terminate if given the input x? This is known as the Halting Problem.

Solution:
Proof. We use a proof by contradiction. Suppose a program $h(p,x)$ succesfully determines whether a program p will halt when given an input x. Let us define a program $k(p)$ which takes the output of $h(p,p)$ (a program ultimately gets encoded as a sequence of bits so we can pass those bits as an input to h) and then halts if $h(p,p)$ returns `runs forever` or runs forever if $h(p,p)$ returns `halts`. In other words, $k(p)$ does the opposite of what $h(p,p)$ does. Well k is just a program like any other, so we can pass it into itself! We have two cases to consider. In the first case, suppose p actually halts, in which case h says it halts, and k then runs forever. However, passing k into itself will call $h(k,k)$, which returns `runs forever`, so k will halt. In the second case, suppose p runs forever, in which case h says it runs forever, and k then halts. However, passing k into itself will call $h(k,k)$, which returns `halts`, so k will run forever. In both of the above cases, we have a contradiction because k would need to run forever and halt, meaning there is no way h could return the correct result. ☐

Which of these are functions?



(a) a function (b) not a function

The technique used in the motivating example is called *diagonalization*, which can also be used to prove that the set of real numbers is what is called *uncountably infinite*. Before being able to count infinities (I know it sounds weird), we need to understand a few properties about functions.

1 Functions: surjective, injective, bijective

You've seen functions before, probably something like $y = x^2$. In particular, we have taken some input *domain* and *mapped* it some other domain (called the *codomain*), using a specific *relation*. Let's now define all of these terms.

Definition 1. *A **function** f is a map from the sets A to B such that every element of A maps to a* unique *element of B. Mathematically, the function f is represented as $f\colon A \to B$, which describes that f maps elements from A to B:*

$$\forall a \in A,\ \exists b \in B \quad \text{such that} \quad \underbrace{f(a) = b}_{\text{image of } a}.$$

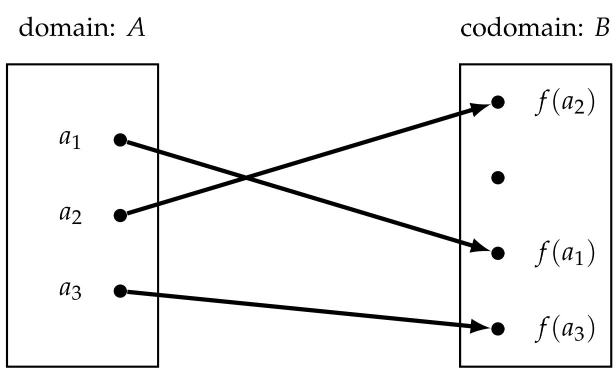
*The set A is referred to as the **domain** and the set B is referred to as the **codomain**.*

codomain versus range?

Note that the range and the codomain are not necessarily equal! The range is a subset of the codomain, since a function may not map a particular input to a value in the codomain. For example, consider the function $f\colon \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x^2$. The domain is $\mathbb{Z}$, the codomain is also $\mathbb{Z}$, but the range is $\{0, 1, 4, 9, 16, \dots\}$, which does not include every element in the codomain $\mathbb{Z}$.

Given a function f and an input a, we say the **image** of a is $f(a) = b$. The set of all images of (for every possible $a \in A$) is called the **range** of f. We also say that the **preimage** of b is a: $f^{-1}(b) = a$.

It is often helpful to visualize how a function maps values from the domain to its codomain using a diagram.



range: $\{f(a_1),\ f(a_2),\ f(a_3)\}$

Example 2:
Which of the following statements is true?

(a) Every relation on A can be expressed as a function $f\colon A \to A$.

(b) Every function $f\colon A \to A$ can be expressed as a relation on A.

Solution:
Only (b) is True. For a set A, a function $f\colon A \to A$ is a subset of $A \times A$. Furthermore, a relation on A is all pairs (a, a) such that $f(a) = a$. The reason (a) is False is because we cannot necessarily find a function that represents a particular relation. For example, consider the relation R on $A = \{0, 1\}$ which is just the Cartesian product of the elements of A: $R = \{0, 1\} \times \{0, 1\} = \{(0, 0),\ (0, 1),\ (1, 0),\ (1, 1)\}$. We cannot find a function that represents this relation because elements of A map to more than one value in R.

Example 3:
Identify the **domain**, **codomain** and **range** of the following functions:

(a) $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2n$.

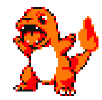
(b) $g\colon \{1,2,3\} \to \{a,b,c\}$ defined by $g(1) = c$, $g(2) = a$ and $g(3) = a$.

Solution:

(a) The domain and codomain are the set of integers. The range is the set of even numbers.

(b) The domain is the set $\{1,2,3\}$. The codomain is the set $\{a,b,c\}$. The range is $\{a,c\}$.

But $g(1) = a$ and $g(3) = a$?

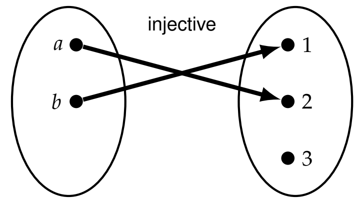


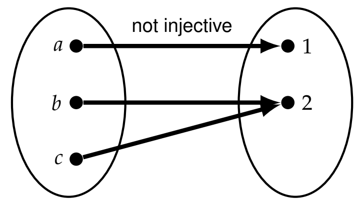
This is okay because in order for g to be a function, we only require that each input maps to a single output.

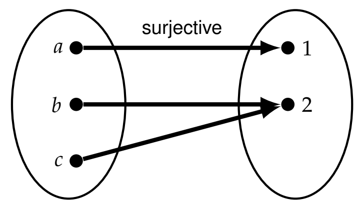
1.1 Properties of functions

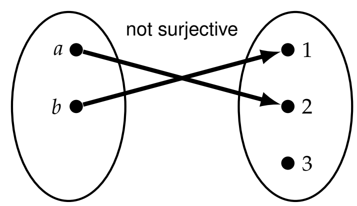
In the last example, we saw that $g(1) = a$ and $g(2) = a$. In other words, g mapped two different inputs to the same output. Functions that do **not** do this are called *injective*. More specifically, injective functions are functions in which *at most one* element of the domain maps to a particular element of the codomain. These are also appropriately called *one-to-one* functions.

If every element of the codomain is the image of some element in the domain, then we say that the function is *surjective*, which is also referred to as an *onto* function. A surjective function is one in which the range and codomain are identical.









Example 4:
Which of the following functions are surjective? Take $\mathbb{R}^+$ as the set of positive real numbers that includes 0.

(a) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

(b) $f\colon \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = x^2$.

Solution:

(a) not surjective because the range is $\mathbb{R}^+$ but the codomain is $\mathbb{R}$.

(b) surjective because both the range and codomain are $\mathbb{R}^+$.

Functions which are both injective and surjective are called **bijective**. These propeties of functions are summarized below. Let $f\colon A \to B$ be a function. Then f is:

- **injective** if for every $x, y \in A,\ x \neq y$ means $f(x) \neq f(y)$.
- **surjective** if for every $b \in B$, there is an $a \in A$ with $f(a) = b$.
- **bijective** if f is both injective and surjective.

sur?

If you speak French, you may know the word *sur* which means "on top of", similar to how we say surjective functions are *onto.*

Example 5:
Let S be the set of all Middlebury students. Determine if the following functions are injective, surjective, bijective or none.

(a) Let E be the set of all Middlebury email addresses.
Define a function $f\colon S \to E$ such that $f(x)$ is x's email address.

(b) Let $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
Define a function $g\colon S \to M$ such that $f(x)$ is x's birth month.

(c) Let $A = 0, 1, 2, \ldots$.
Define a function $h\colon S \to A$ such that $f(x)$ is x's age in years.

Solution:
(a) injective because no two people have the same email address.

(b) surjective because some student is born in every month.

(c) not surjective because no students are 1000 years old and not injective because two students can have the same age.

2 Counting infinite sets (optional)

This might sound weird, but we can actually "count infinities." The cardinality of the natural numbers is $|\mathbb{N}| = \infty$. A valid question is: what is the cardinality of the set of even numbers $|E|$? Is it the same infinity?

For sets A and B, the cardinalities of both sets are the same if we can find a bijective function that maps every element of a to an element of B. Sets which have the same cardinality as the set of natural numbers are called **countably infinite**. In the case of the even numbers, we can define a bijective function $f\colon \mathbb{N} \to E$ with $f(x) = 2x$. Therefore, $|E| = |\mathbb{N}|$ and E is countably infinite.

Example 6:
Consider the set $S = \{x \in \mathbb{Z}\colon x > 10\}$. Is $|S| = |\mathbb{N}|$?

Solution:
We can determine if S is countably infinite by finding a bijective function from $\mathbb{N}$ to S. We can do this with the function $f\colon \mathbb{N} \to S$ such that $f(x) = x + 10$.

Sets which are larger than the set of natural numbers are called uncountably infinite. To show that a set is uncountably infinite, we just need to show that we cannot find a bijective function from $\mathbb{N}$ to the particular set. A famous set that is uncountably is the set of real numbers $\mathbb{R}$. We can show that $\mathbb{R}$ is uncountably infinite by using a diagonalization argument similar to how we proved that it is impossible to write a program to determine if a program halts or runs forever.

How do I show it is uncountably infinite?

We need to show that there does not exist a bijective function from $\mathbb{N}$ to the set in question. Remember that a **proof by contradiction** is useful to show that something *does not exist*. Start by assuming that a bijective function does exist, and then show we ultimately end up with a contradiction.

Example 7:
Prove that the set of real numbers is uncountably infinite.

Solution:
Proof. We use a proof by contradiction. Suppose $\mathbb{R}$ is countable. Then any subset of $\mathbb{R}$ is also countable. Consider the set of real numbers $\mathbb{R}_1$ between 0 and 1 and note that $\mathbb{R}_1 \subset \mathbb{R}$.

Assume there is a bijective function $f\colon \mathbb{N} \to \mathbb{R}$. Let us represent the i^{th} real number r_i by $0.d_{i1}d_{i2}d_{i3}\ldots$ where d_{ij} is the j^{th} decimal place of the i^{th} real number. For example,

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}\ldots \\ &\vdots \end{aligned}$$

where $d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$. Note that the number $\frac{1}{2}$ would be represented as $0.500000\ldots$ (the 0's never end). Now form a new real number (which should be in $\mathbb{R}_1$) with the following rule:

$$r = 0.d_1d_2d_3d_4\ldots \qquad d_k = \begin{cases} 4 & \text{if } d_{kk} \neq 4 \\ 5 & \text{if } d_{kk} = 4. \end{cases}$$

For example, starting with

$$\begin{aligned} r_1 &= 0.\mathbf{1}5434\ldots \\ r_2 &= 0.4\mathbf{1}198\ldots \\ r_3 &= 0.97\mathbf{4}22\ldots \\ &\vdots \end{aligned}$$

we would get the number $r = 0.445\ldots$. However, note that r is not on the list! Every number in $\mathbb{R}_1$ has a unique expansion, but r cannot be in $\mathbb{R}_1$ because r differs from any possible r_i in the i^{th} decimal place. Therefore, it is not possible to represent all real numbers between 0 and 1, so we have a contradiction. $\square$

This is known as *diagonalization* because of the way we change the diagonal entries of the decimal places. It was first used to prove that the real numbers are uncountably infinite in 1891 by Georg Cantor.