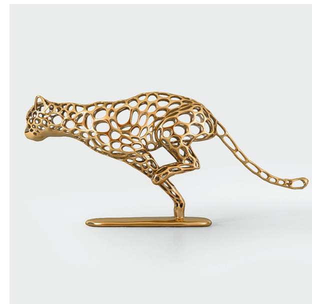
Learning objectives:

- ☐ describe (in words) what a relation is,
- ☐ describe three properties of relations (reflexive, symmetric, transitive) and how this gives us equivalence relations,
- ☐ describe how equivalence classes partition a set,
- ☐ prove whether (or not) a relation is an equivalence relation.

3d printed cheetah



optimization of giraffe patterns

The ultimate goal of today's lecture is to divide a set into subsets, using some mathematical notation. We may want to do this because we only want to work with subsets that share a similar property. In particular, if we are writing an algorithm or proving something, it might be easier to work with these subsets instead of the full set. Before we do that, a little exercise.

Example 1:

Suppose I give you a set of points Z in the plane ($\mathbb{R}^2$). Our goal, is to *partition* the plane in such a way that the *regions* contain all points of the plane *closest* to a particular point in Z. For example, for a point $\vec{z}_i \in Z$, the region $R_i \subset \mathbb{R}^2$ is defined such that $d(\vec{z}_i, \vec{x}) \leq d(\vec{z}_j, \vec{x})$, $\forall \vec{x} \in \mathbb{R}^2$, $\forall \vec{z}_j \in Z$, $i \neq j$. Note the use of the Euclidean distance d. For a bunch of random points on the board (Z), see if you can draw the partitions of the board.

In case you're interested, this creates what is called a *Voronoi* diagram, which shows up naturally in the patterns on giraffes, the skeleton structure of dragonfly wings, creases in drying soil and many more! Modern artists are also finding uses for Voronoi patterns when creating 3d printed models (see the cheetah on the right).

We'll come back to this at the end of the lecture. For now, consider some simpler examples.

Example 2:

Let A be the set of all cities in the United States and let B be the set of all states. The Cartesian product of A and B, denoted by $A \times B$ is the set of pairs:

$$A \times B = \{(\text{Montpelier, AK}, \ \text{Montpelier, ME}, \ \text{Montpelier, MA}, \ \ldots, \ \text{Montpelier, VT}, \\ \text{Boston, AK}, \ \text{Boston, ME}, \ \text{Boston, MA}, \ \ldots, \ \text{Boston, VT}, \ \ldots\}.$$

Now, only a subset of this product correspond to the actual capital cities paired with the appropriate states.

Consider a more mathematical example:

Example 3:
Consider the mod function, specifically $\text{mod}(x, 4)$, $\forall x \in \mathbb{Z}$. Note how this function partitions $\mathbb{Z}$ into four distinct sets:

$$M_0 = \{\dots,\ -12,\ -8,\ -4,\ 0,\ 4,\ 8,\ 12,\ \dots\}$$
$$M_1 = \{\dots,\ -11,\ -7,\ -3,\ 1,\ 5,\ 9,\ 13,\ \dots\}$$
$$M_2 = \{\dots,\ -10,\ -6,\ -2,\ 2,\ 6,\ 10,\ 14,\ \dots\}$$
$$M_3 = \{\dots,\ -9,\ -5,\ -1,\ 3,\ 7,\ 11,\ 15,\ \dots\}$$

In other words, we've separated the set of integers $\mathbb{Z}$ into four sets, depending on whether an integer has modulus of 0, 1, 2 or 3 with the number 4. In general, the modulo n function will separate the integers into n distinct sets.

Now, let's describe how to characterize these subsets mathematically.

1 Relations

Definition 1. *Let A be a set. A **relation** R on A is a subset of $A \times A$.*

$$R \subseteq A \times A$$

If $(x, y) \in R$, then we say xRy or x *is related to* y.

Cartesian product?

The **Cartesian product** of two sets A and B is the set of *ordered pairs* of elements in A with those in B:

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

The lecture on sets has more information and examples on the Cartesian product.

Example 4:
Let R_1 be the relation $\{(x, y) \mid x < y\}$. Which of the following are in R?

(a) $(1, 2)$

(b) $(2, 1)$

(c) $(1, 1)$

Solution:
$(1, 2) \in R_1$ since $1 < 2$, however, none of the others are in R_1 because the order matters. If instead, we considered the relation $R_2 = \{(x, y) \mid x \leq y\}$ then option (c) is in R since $1 \leq 1$ so $(1, 1) \in R_2$.

When we considered $(1, 1) \in R_2$ in the last example, this is actually a special property of relations.

2 Properties of relations

There are three properties of relations that we want to look at. Let A be a set and R be a relation, $R \subseteq A \times A$.

- **reflexive:** $(a, a) \in R$, or aRa, $\forall a \in A$ (example: $\leq$).
- **symmetric:** if $(a, b) \in R$, then $(b, a) \in R$, $\forall a, b \in A$ (example: $ab \geq 1$, $ba \geq 1$).
- **transitive:** if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, $\forall a, b, c \in A$. (example: $<$ because $a < b$ and $b < c$ means $a < c$).

Now we can define **equivalence relations**.

Definition 2. *An **equivalence relation** R on the set A is reflexive, symmetric and transitive.*

How to show if something is an equivalence relation?

To show a relation is an equivalence relation, just check the three properties! If you want to show that something is *not* an equivalence relation, finding a counterexample is useful.

Example 5:
Let's return to our modulus example. Instead of looking at the special case of modulo 4, we'll look at modulo m for any integer m. The relation can be described by

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

Note that $a \equiv b \pmod{m}$ means m *divides* $a - b$. Is R an equivalence relation?

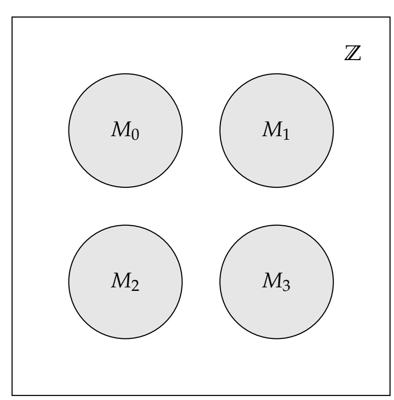
Solution:
To check if R is an equivalence relation, we need to check the three properties.

- **reflexive?** $a - a = 0$ and m certainly divides 0. So R is reflexive.
- **symemtric?** if m divides $a - b$, then $a - b = km$, $k \in \mathbb{Z}$. We need to check if m divides $b - a$. $b - a = -(a - b) = -km$. So R is symmetric.
- **transitive?** if m divides $a - b$ and m divides $b - c$, then does m divide $a - c$? If m divides $a - b$, then $a - b = km$, $m \in \mathbb{Z}$. If m divides $b - c$, then $b - c = lm$, $l \in \mathbb{Z}$. Then $a - c = (a - b) + (b - c) = km + lm = (k + l)m$. Since $(k + l) \in \mathbb{Z}$, then R is transitive.

Since we verified all three properties, then R is an equivalence relation.

Notice how an equivalence relation **partitions** a domain into **equivalence classes**. If a pair $(a, b) \in R$, then the notation $a \equiv b$ means that a and b are in the same equivalence classes. The diagram on

Equivalence classes for the relation n mod 4.



the right demonstrates how congruence modulo m partitions the set of integers ($\mathbb{Z}$) into m equivalence classes (for the special case when $m = 4$). We can also go the other way around. Given a partition of a set $A = \{A_i \mid i = 1, 2, \ldots\}$, there exists an equivalence relation R that has sets A_i, $i = 1, 2, \ldots$ as it's equivalence classes. Remember the giraffe patterns from the beginning of the lecture. Well, in that example we partitioned the board ($\mathbb{R}^2$) into the different regions. Does this partition define an equivalence relation?

Example 6:
Check whether the partition we found at the start of the lecture defines an equivalence relation.

Solution:
In order to check if this is an equivalence relation, we need to check the three properties.

reflexive? we need to check that $(a, a) \in R$, which is true since a is in the same subset as itself!

symmetric? if $(a, b) \in R$, then is $(b, a) \in R$? This is also true since this means that both a and b are in the same region.

transitive? if $(a, b) \in R$ and $(b, c) \in R$, then is $(a, c) \in R$. In other words, given that a and b are in the same region, and b and c are in the same region, then are a and c in the same region? This is also true, because all three are in the same region!

So we've done a few examples to show if something is an equivalence relation, but how do we show if something is *not* an equivalence relation? The easiest method is to use a **proof by counterexample**.

Example 7:
Show that the relation $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \colon a \mid b\}$ is not an equivalence relation.

Solution:
We can find an example to show that the relation is not symmetric: $2 \mid 4$ but $4 \nmid 2$.

Example 8:

Let S be the set of all people. Decide if the following is an equivalence relation or not, and explain why. If it is an equivalence relation, what are the equivalence classes?

(1) $R \subseteq S \times S, R = \{(a, b) \colon a, b \text{ have same parents}\}$.

(2) $R \subseteq S \times S, R = \{(a, b) \colon a, b \text{ share a parent}\}$.

Solution:

(1) It is an equivalence relation. The equivalence classes are the groups of full siblings.

(2) It is not an equivalence relation because the transitive property is not satisfied. This can be verified with the following family tree.