Goals for today:

1. List items in the **Cartesian product** of two sets.
2. Describe (in words) what a **relation** is.
3. Describe three properties of relations: **reflexive, symmetric, transitive**.
4. **Partition** a set into equivalence classes.
5. Show whether or not a relation is an equivalence relation.

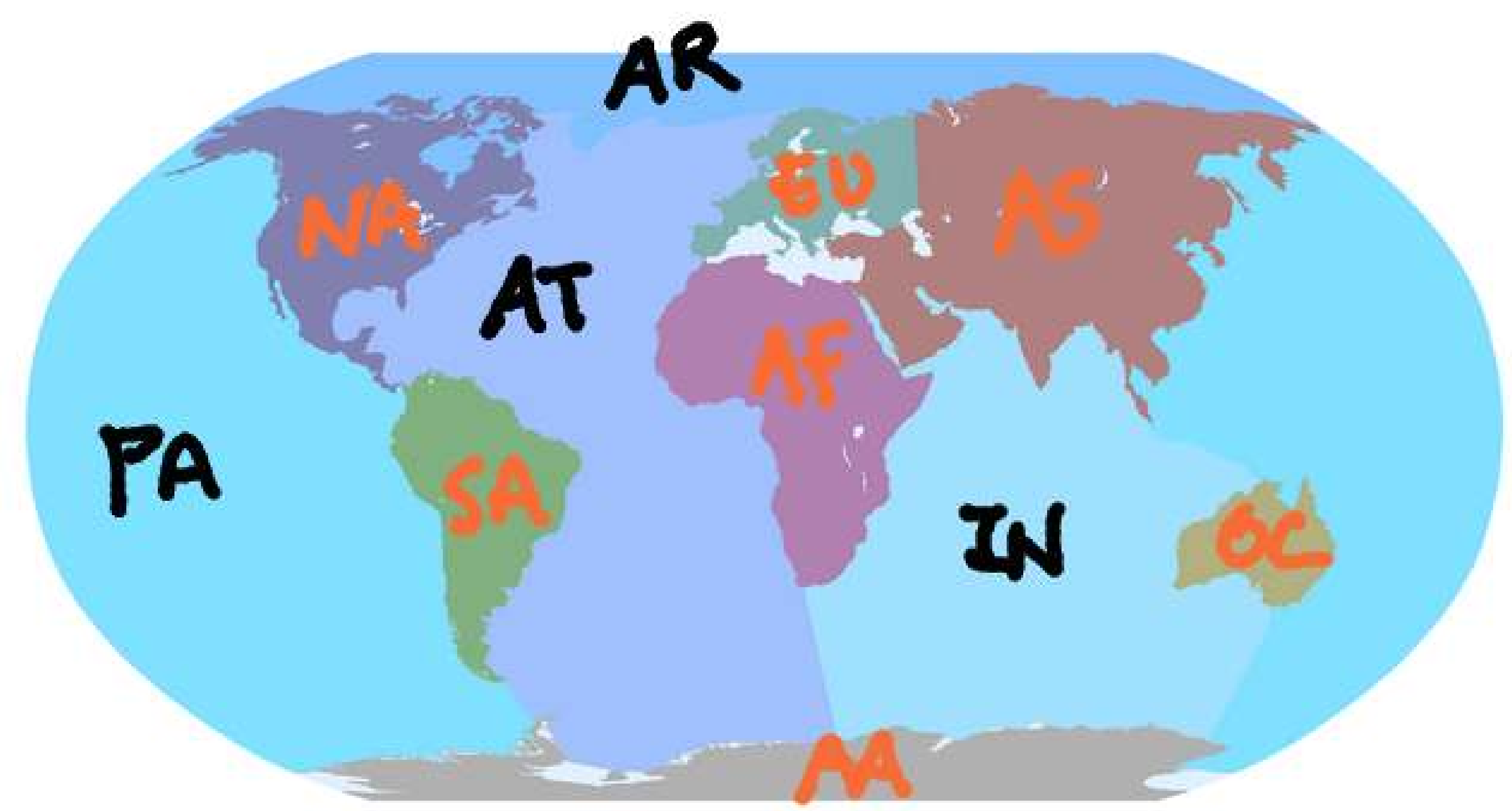
→ partition into "cells"

Next few course topics might feel random, but I promise everything is "related"!

The Cartesian Product of two sets: $A \times B$.

$A = \{PA, AT, AR, IN\}$

$B = \{NA, SA, AF, EU, AS, OC, AA\}$



$A \times B = \{(PA, NA), (PA, SA), (PA, AF), (PA, EU), \ldots, (AT, NA), \ldots\}$

	NA	SA	AF	EU	AS	OC	AA
PA	1	1	0	0	1	1	1
AT	1	1	1	1	0	0	1
AR	1	0	0	1	1	0	0
IN	0	0	1	0	1	1	1

what if we only want to extract ocean-continent pairs that border each other?

extract a subset from $A \times B$

A *relation* R on a set A is a subset of $A \times A$, filtered by some predicate.

$R \subseteq A \times A$

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Let R1 be the relation on Z: R1 = { (x, y) : x < y }. Which of the following are in R1? 35

- [x] (1, 2)
- [ ] (2, 1)
- [ ] (1, 1)

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Let R2 be the relation on Z: R2 = { (x, y) : x ≤ y }. Which of the following are in R2? 40

- [x] (1, 2)
- [ ] (2, 1)
- [x] (1, 1)

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In the last example, R_2 satisfied a certain property - what are properties of relations we might care about?

let A be a set, and R is a relation on A, $R \subseteq A \times A$.

- **Reflexive:** $(a,a) \in R \quad \forall a \in A$ eg. $\leq$
- **Symmetric:** $(a,b) \in R$ $\forall a,b \in A$ eg. $ab \leq 1$

  $(b,a) \in R$ $ba \leq 1$
- **Transitive:**

  $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ $\forall a,b,c \in A$ eg. $<$ $a < b$, $b < c$ $a < c$? yes.

a relation that satisfies all 3 properties is an equivalence relation.

An *equivalence relation* is a relation that is (1) reflexive, (2) symmetric and (3) transitive.

Example: Is $R = \{(a,b) : \underbrace{a \equiv b \pmod{m}}_{\text{means } a-b \text{ is an integer multiple of } m.}\}$ an equivalence relation?

reflexive? check if $a-a$ is an int. multiple of m? $a-a=0=0\cdot m$ ✓

symmetric? if $(a,b)\in R \to$ then $a-b=km$, $k\in\mathbb{Z}$

is $(b,a)\in R$? $\to$ check if $b-a$ = int. mult. of m?

$b-a=-(a-b)=(-k)m$ ✓

transitive? $(a,b)\in R$ $\quad a-b=km \quad k\in\mathbb{Z}$

and $(b,c)\in R$ $\quad b-c=\ell m \quad \ell\in\mathbb{Z}$.

check if $(a,c)\in R$. $\quad a-c=a-b+b-c$

$= km+\ell m = \underbrace{(k+\ell)}_{\text{int}} m$ ✓

yes, it's an equivalence relation.

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Example: Is $R = \{(a, b) : a \equiv b \pmod{m}\}$ an equivalence relation?

what does this mean about original set?

let $m = 4$

possible values of % 4

0, 1, 2, 3

$\{\dots, -4, 0, 4, 8, \dots\}$

M_0

$\{\dots, -3, 1, 5, 9, \dots\}$

M_1

partitioned set into classes. (equivalence classes).

$\{\dots, -2, 2, 6, 10, \dots\}$

M_2

$\{\dots, -1, 3, 7, 11, \dots\}$

M_3

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Why is $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : \underbrace{a \mid b}_{\text{divides.}}\}$ not an equivalence relation?

reflexive? $a \mid a$? ✓

symmetric? if $a \mid b$ does $b \mid a$?

counterexample

e.g. $2 \mid 4$ but $4 \nmid 2$

↑ does not divide.

One last example!

Let S be the set of all people. Which of the following are equivalence relations? Why (or why not)? If they are equivalence relations, what are the equivalence classes?

equivalence classes: groups of siblings. ✓

1. $R \subseteq S \times S,\ R = \{(a, b) :\ a, b$ have the same parents $\}$

reflexive? ✓

symmetric? ✓

transitive? $(a,b) \in R$, $(b,c) \in R$ $\Rightarrow$ $(a,c) \in R$?

2. $R \subseteq S \times S,\ R = \{(a, b) :\ a, b$ share a parent $\}$

reflexive? ✓

symmetric? ✓

transitive? ✗