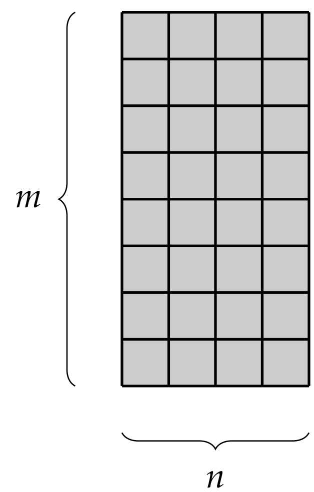
Learning objectives:

- ☐ identify when to use strong induction versus ordinary induction
- ☐ identify when multiple base cases are needed in a proof by induction

We've been practicing a lot with induction so far. We've restricted our attention to *ordinary induction*, in which the inductive step was proved for $p(n+1)$, assuming that $p(n)$ is true (where $p(n)$ is some predicate). Today, we'll look at *strong induction*. The main difference with strong induction is that, instead of merely assuming $p(n)$ is true to prove $p(n+1)$, we'll assume all $p(0),\ p(1),\ p(2),\ \ldots,\ p(n)$ are true. But first a challenge!

Example 1:
Suppose you have a chocolate bar with $m \times n$ squares. We want to break up the chocolate bar into individual squares. You can split the chocolate bar by rows or columns, repeatedly, and eventually break up the entire bar. Prove that the number of "breaks" you make does not depend on the sequence of the breaks.

Hint: try and come up with the exact number of breaks you need.



1 Principle of strong induction

Let p be a predicate. If

- $p(0)$ is true,
- $p(0) \wedge p(1) \wedge p(2) \wedge \cdots \wedge p(n) \implies p(n+1)$,

then $p(n)$ is true for all n. The only difference between ordinary induction and strong induction is that you can use more information when proving the inductive step. In particular, you can assume that the induction hypothesis holds for all cases "lower" than the $n+1$ case you are trying to prove. Let's go back to our chocolate bar example to see where strong induction is useful.

Base case?

Here, $n = 0$ is used as the base case. However, this definition should be understood as $p(\text{base})$, and subsequently all $p(\square)$ for some $\square$ "higher" than the base case.

Solution:
As suggested in the hint, it's easier to try to prove the actual number of breaks is equal to some specific value, regardless of the sequence of the breaks. The trick with this question is that you might get thrown off by the mn and think there are two variables the control the number of breaks. In fact, it's easier to think about this problem in terms of the *total* number of squares.

Theorem 1. *We need $mn - 1$ breaks to split up a chocolate bar with $m \times n$ pieces.*

Proof. We use a proof by strong induction on the total number of squares in the chocolate bar. Let $p(k)$ be the predicate that a chocolate bar with k squares requires at most $k - 1$ breaks.

Base case: when $k = 1$, there is only one square, so no breaks are needed. Thus at most $1 - 1 = 0$ breaks are needed.

Inductive step: Suppose $p(1) \wedge p(2) \wedge \cdots \wedge p(k)$ are true. This means that any chocolate bar with $1 \leq s \leq k$ squares requires at most $s - 1$ splits. We must show that a chocolate bar with $k + 1$ squares requires at most k breaks.

Since $k > 1$, we can break this chocolate bar into two pieces, incurring a single break. Now, suppose that the first piece has a squares, and the second piece has b squares. Then by our assumption, the piece with a squares requires at most $a - 1$ breaks and the piece with b squares requires at most $b - 1$ breaks. The total number of splits to break the chocolate bar with $k + 1$ squares is at most $a - 1$ (piece with a squares) $+b - 1$ (piece with b squares) $+1$ (breaking the $k + 1$ bar into two pieces) $= a + b - 1$. Since $a + b = k + 1$, then the total number of breaks is $k + 1 - 1 = k$.

Therefore, by strong induction on the number of squares in a chocolate bar, $p(k)$ is true. □

As a result, a bar with mn squares requires at most $mn - 1$ breaks.

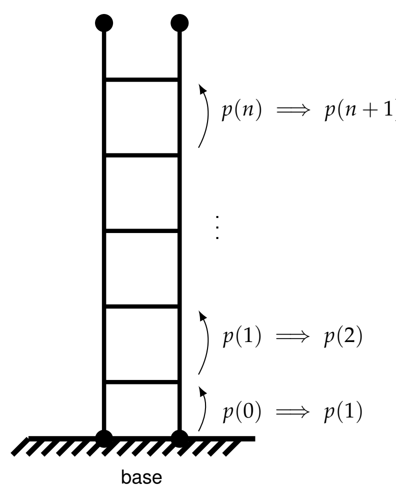
Example 2:
Prove the following theorem.
Theorem 2. *Every integer greater than* 1 *is either a prime number or is the product of prime numbers.*

Proof. We will prove the theorem by strong induction on integers $n > 1$. Let the induction hypothesis be $p(n) =$ *an integer greater than* 1 *is either prime or a product of primes*.

Base case: For $n = 2$ we have that 2 is a prime number.

Inductive step: Let $n \geq 2$ and assume that $p(2) \wedge p(3) \wedge \cdots \wedge p(n)$ are true. We need to show that $p(n+1)$ is either a prime number or a product of primes. There are two cases to consider: (1) $n+1$ is already prime (in which there is nothing to show), or (2) $n+1$ is not prime. Let $n+1$ not be a prime number. This means $n+1$ can be factored into $n+1 = k \times m$ for some integers k and m between 2 and n. By our assumption about $p(k)$ and $p(m)$, we know that k and m are either prime or the product of primes. In either case, since $n+1$ is the product of k and m, which is a product of prime numbers, then $n+1$ would be the product of prime numbers.

Therefore, by strong induction on n, any integer $n > 1$ is either a prime number or a product of prime numbers. □



2 Multiple base cases

Sometimes a single base case is not enough. Think back to the ladder metaphor. When climbing a ladder, you don't really jump both feet one rung at a time, do you? You usually have, say your left foot on a particular rung, then your right foot on the one above. So once you start climbing the ladder, you're using both your feet as your base. When doing a proof by induction, you will sometimes need more than one foot as your base. It will often be clear, based on the expression you are trying to prove, when you will need several base cases.

Example 3:
Consider the following recurrence relation on the positive integers:

$$F(n) = 4F(n-1) - 4F(n-2) \tag{1}$$

with $F(0) = 1$ and $F(1) = 0$.

Prove that $F(n) = 2^n(1-n)$ for $n \geq 0$.

Solution:

Proof. We use strong induction on the integer n. Let the induction hypothesis be $p(n) = F(n) = 2^n(1-n)$.

Base cases:

For $n = 0$, $F(0) = 2^0(1-0) = 1$.

For $n = 1$, $F(1) = 2^1(1-1) = 0$.

Inductive step: Assume $p(0) \wedge p(1) \wedge \cdots \wedge p(n)$ are true. That is, $F(k) = 2^k(1-k)$ for $0 \leq k \leq n$. We need to show that $F(n+1) = -2^{n+1}n$. By the recurrence relation of Equation 1,

$$\begin{aligned} F(n+1) &= 4F(n) - 4F(n-1) \\ &= 4 \cdot 2^n(1-n) - 4 \cdot 2^{n-1}(1-(n-1)) \\ &= 4 \cdot 2^{n-1}\left(2(1-n) - (2-n)\right) \\ &= 2^2 \cdot 2^{n-1}(-n) \\ &= -2^{n+1}n \end{aligned}$$

Therefore, by strong induction on n, $p(n)$ is true. □

These types of problems show up a lot when solving recurrence relations, which we will see later in the course.