goals:

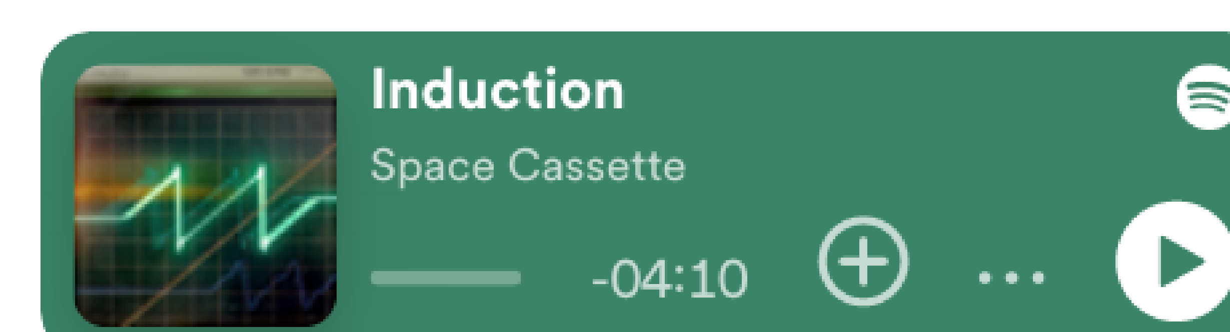
- identify when to use strong induction
- identify when to prove multiple base cases

Middlebury

CSCI 200: Math Foundations of Computing

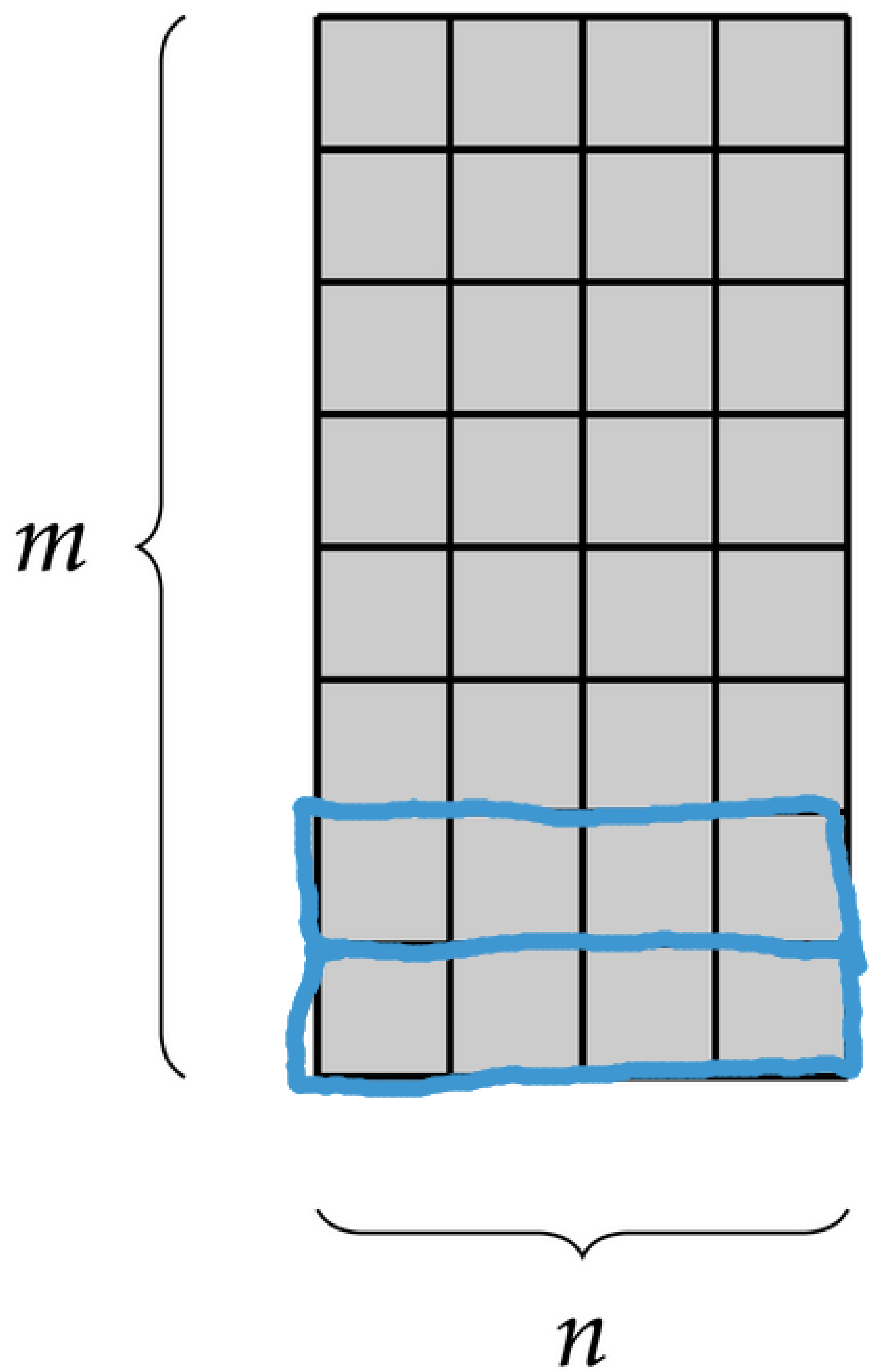
Spring 2024

Lecture 4M: Strong Induction



How many "breaks" to break up $m \times n$ chocolate bar into individual squares?

$m \cdot n^2$ or $m^2 n$



$\#$ squares -1

theorem: It takes $mn - 1$ breaks to break up a $m \times n$ chocolate bar into individual squares.

proof: soon!

The Principle of Strong Induction

Let p be a predicate. If

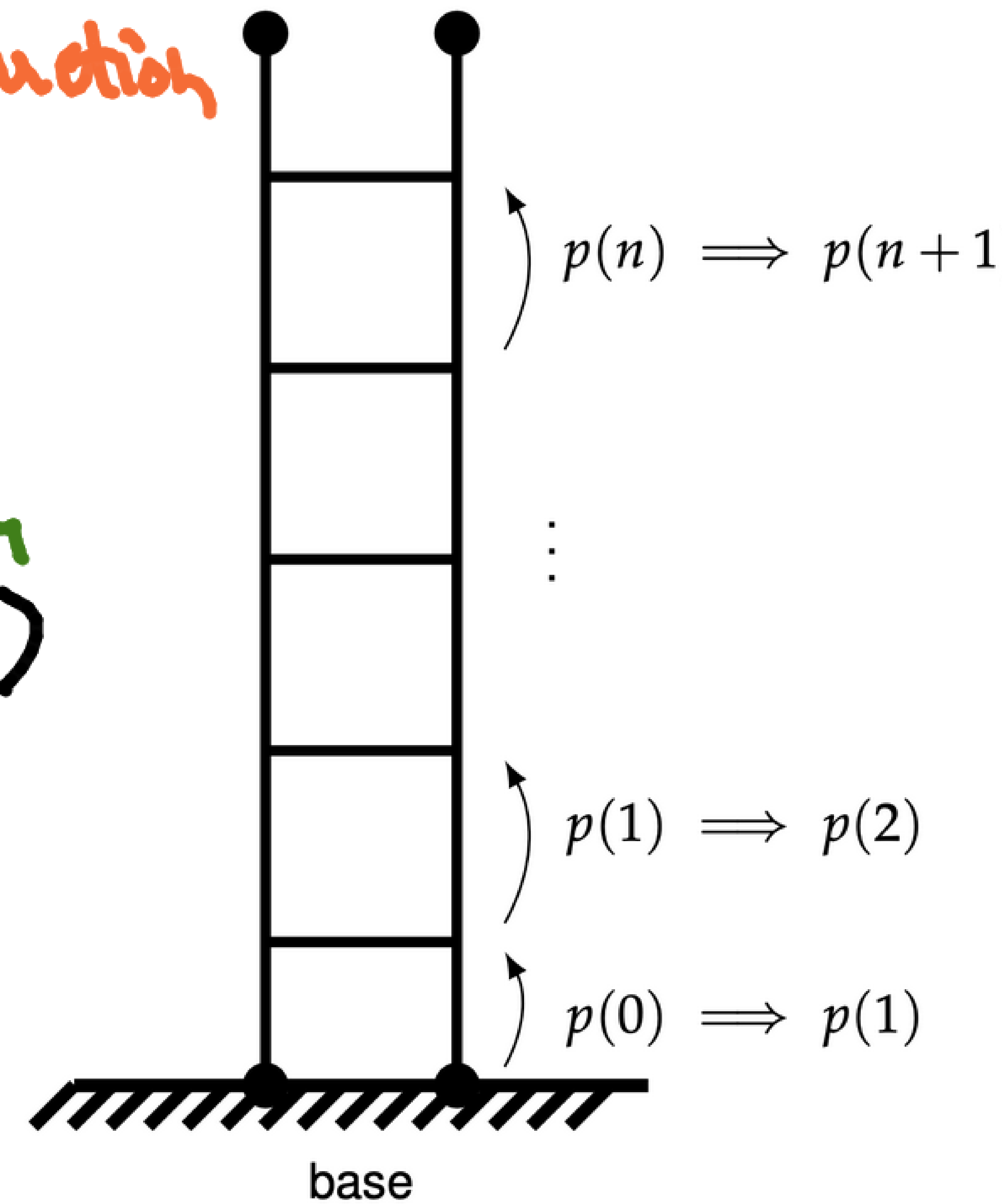
→ whatever the lowest value of induction variable is.

$p(0)$ is true

AND

$$\overbrace{p(0) \land p(1) \land p(2) \land \ldots\ldots \land p(n-1) \land p(n)}^{\text{assume this}} \xrightarrow[\text{implies}]{} p(n+1)$$

then $p(n)$ true for all n.



Prove that it takes $mn - 1$ "breaks" to break up the $m \times n$ chocolate bar into individual squares.

What should our induction variable be? 22

something else
95%

n
5%

m
0%

Edit response

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let $k = mn$
be the #
squares.

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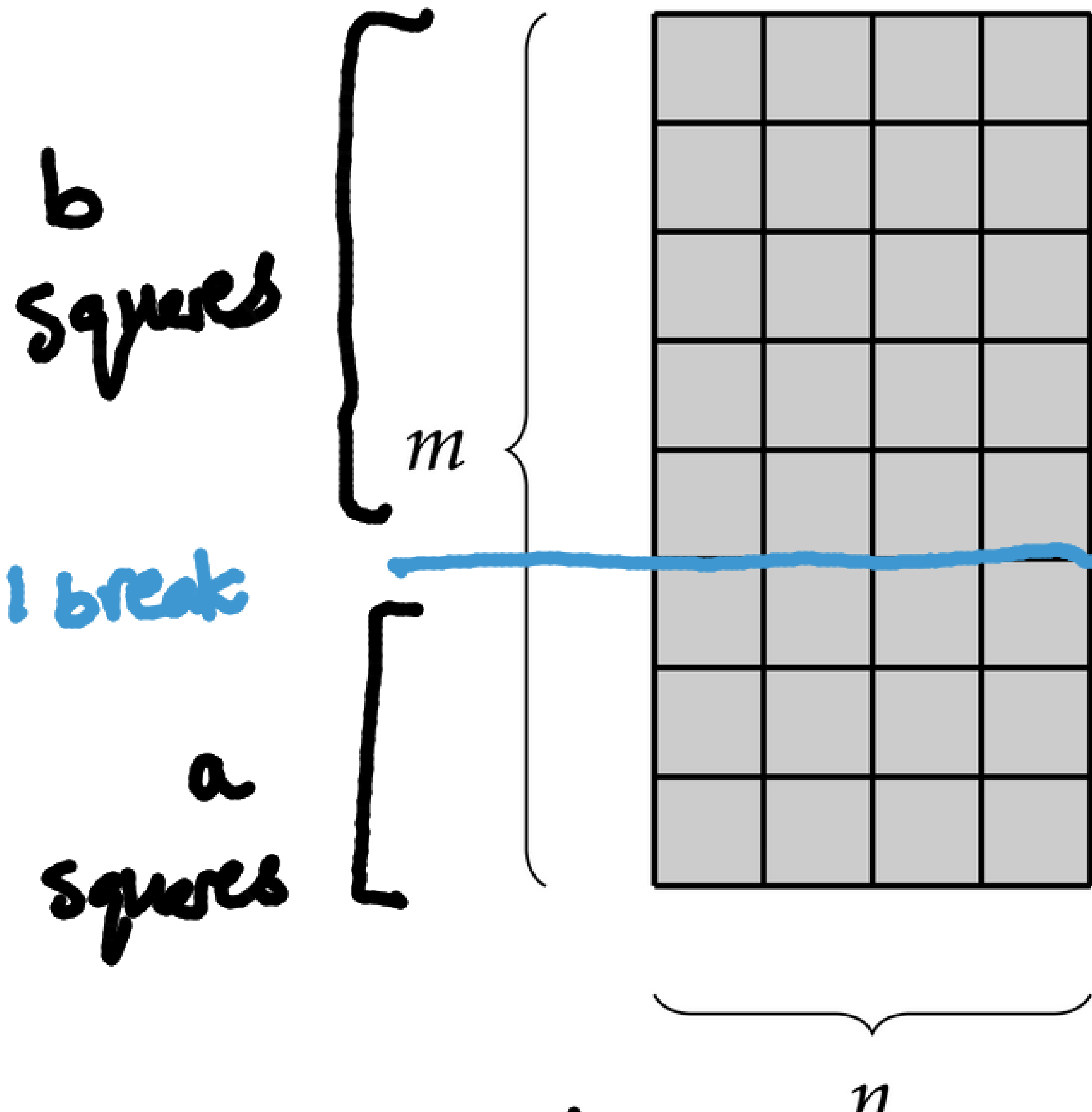
Prove that it takes $mn - 1$ "breaks" to break up the $m \times n$ chocolate bar into individual squares.



__proof:__ We use a proof by strong induction on the number of squares k. Let the IH be the predicate $p(k)$ = "it takes $k-1$ breaks to break up a chocolate bar with k squares."

__base case:__ For $k=1$, we have a bar with a single square, which takes 0 breaks and our formula says it should take $1-1=0$ breaks. ✓

__inductive step:__ Assume $p(1) \wedge p(2) \wedge \ldots \wedge p(k-1) \wedge p(k)$ true.

We will show $p(k+1)$ true, i.e. it takes $k+1-1$ breaks. (k).

Let's start with a $(k+1)$ bar, now break into two pieces with a and b squares. We know (by assumption), it takes $a-1$ and $b-1$ breaks to break up each bar, so the total number of breaks is

$$a-1+b-1+1 = \underbrace{a+b}_{k+1}-1 = k+1-1 = k.$$

Therefore, by strong induction on k, $p(k)$ is true. □

Example 3: Recurrence relations.

Consider the following "recurrence relation":

$$F(n) = 4F(n-1) - 4F(n-2), \quad \overbrace{F(0) = 1,\ F(1) = 0.}^{\text{multiple base cases (need to prove)}}$$

We want a "closed-form" expression for $F(n)$ that doesn't depend on $F(n-1), F(n-2)$, etc. **Prove that** $\underbrace{F(n) = 2^n(1-n)}$ **for all** $n \geq 0$.

Start with recurrence relation,

assume $P(0) \wedge P(1) \wedge \ldots \wedge P(n-1) \wedge P(n)$

show $P(n+1)$