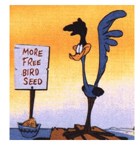
Learning objectives:

- ☐ state the basic steps of a proof by induction
- ☐ prove some simple propositions using induction

We have seen a variety of methods for proving propositions and implications. Sometimes we need to prove a proposition that involves a set or a sequence of objects. We might also need to prove the correctness of a recursive algorithm. In general, we might need to consider the size of the problem, and show that a proposition or algorithm works regardless of the problem size. This is where induction will be useful. Consider the following motivating example.

Seeds?



1,000,000 seeds is a heap of seeds.
If 1,000,000 seeds is a heap,
then 999,999 seeds is a heap.
So 999,999 seeds is a heap.
If 999,999 seeds is a heap,
then 999,998 seeds is a heap.
So 999,998 seeds is a heap.
⋮
So 1 seed is a heap.

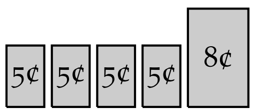
This is known as Sorites Paradox.

Example 1:

Suppose you have unlimited 5¢ and 8¢ postage stamps.

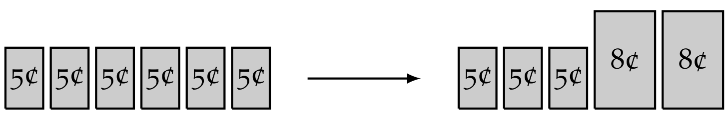
(a) Can you make 4¢? **no!**

(b) Can you make 28¢?

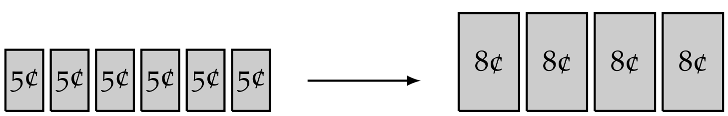


(c) Can you make 85,694¢?

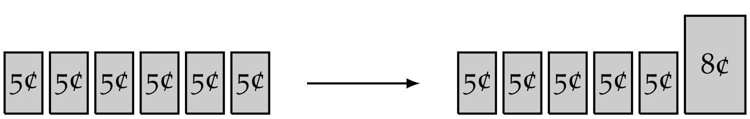
We'll get back to part (c). First, think about if you can make 31¢ starting from 30¢.



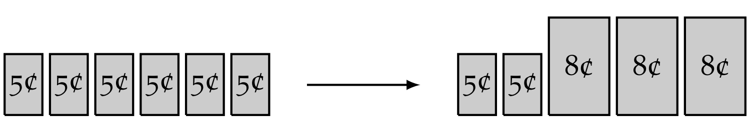
Hmmm, okay, what about 32¢ from 30¢?



And now, 33¢ from 30¢:

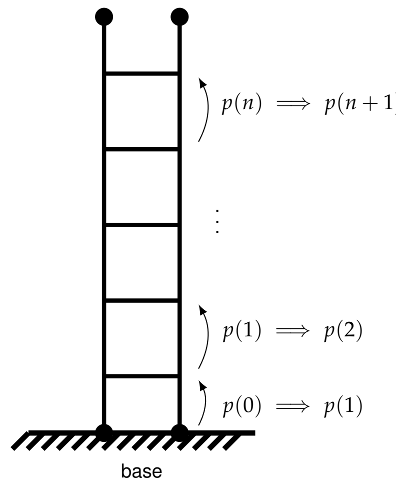


And finally, 34¢ from 30¢:



Clearly, we can make 35¢ from 30¢ by just adding another 5¢ stamp. Wow! So what does this tell us? Well, it means that we can make any postage stamp value (greater than 28) from an existing one. This is the idea behind induction, which we will now look at more formally.

Back to part (c) though. How do you make 85,694? Well, we can make $n = 85,690$¢ with $17,138 \times 5$¢ stamps. We just showed that we can make $(n+4)$¢ by removing 4×5¢ stamps and adding 3×8¢ stamps, so $17,134 \times 5$¢ stamps and 3×8¢ stamps. Note: there are other ways to make this value.



1 The Principle of mathematical induction

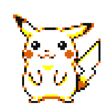
The main idea behind induction is to prove some predicate p holds at $n+1$ if we already know that it holds for some value at n. That is, we want to show that $p(n) \implies p(n+1)$.

Mathematical induction is often related to falling dominoes or climbing a ladder. In the ladder metaphor, you need some kind of *base* to set your ladder on (usually the ground). Now, to get to a particular rung, you need to set yourself on the rung below it. And again, you need to set yourself on the rung below that one, and so on and so forth, until you hit the base.

There are some really important ingredients you need when doing a proof by induction:

- Write: *We use a proof by induction.*
- Identify and state your **induction variable:** this is extremely important and can often be the culprit in a faulty induction proof.
- Identify and state your **induction hypothesis**: this is the predicate that you will suppose on your induction variable. It will often be clear what your induction hypothesis is based on the proposition you are trying to prove.
- Prove your **base case**. Always (always, always) prove your base case. It is also really important to identify the value of your induction variable at the base case.
- Prove the **inductive case**. Prove that $p(n) \implies p(n+1)$. You can use any method we have seen so far.
- **Conclude** your proof by stating *By induction, this means ...*.

How do I approach the inductive case?



There are two main approaches you can use to prove the inductive cases:

1. Start with $p(n)$ = true, manipulate into showing $p(n+1)$ is true. This is common when proving propositions involving some sequence of numbers.
2. Start with $p(n+1)$ and "break off" part of the problem so that you are left with a $p(n)$ part (which you know is true), then bring the extra "+1" part back in. This is more common when proving propositions related to algorithm correctness or with graphs (as we will see later).

However, keep in mind that it is *super incorrect* to prove $p(n+1) \implies p(n)$!

Example 2:
Identify the (a) **base case**, (b) **induction variable** and (c) **induction hypothesis** for the earlier postage stamp example.

Solution:

(a) The base case is for $n = 28$ since this was mentioned in the problem.

(b) The induction variable is n: the value of the postage stamp to be created.

(c) The induction hypothesis is: a postage stamp value of n¢ can be created from 5¢ and 8¢ stamps.

Example 3:
Prove that you can make any stamp value greater than or equal to 28¢ using either 5¢ stamps and 8¢ stamps.

Proof. We use a proof by induction on the stamp value n. Let the induction hypothesis be: $p(n) =$ *you can make a stamp value greater than or equal to* 28 *using either* 5¢ *stamps and* 8¢ *stamps.*

Base case: We can make a value of 28¢ from 4×5¢ stamps and 1×8¢ stamp.

Inductive step: Suppose $p(n)$ is true. That is, we can create a postage value of n with 5¢ stamps and 8¢ stamps. Note that we **must** use at least 3×5¢ stamps **or** 3×8¢ stamps (since using 2 of each would only give a value of 26¢). Therefore, any value of $n + 1$ stamps can be created by either (1) removing 3×5¢ stamps and adding 2×8¢ stamps **or** (2) removing 3×8¢ stamps and adding 5×5¢ stamps. Therefore, any value of $(n + 1)$¢ can be made from n¢.

By the principle of mathematical induction, $p(n)$ is true for all $n \geq 28$. □

2 Examples

Example 4:
Prove $2^n - 1 \leq 3^n$ for all integers $n \geq 0$.

Solution:
Proof. We use a proof by induction on an integer n. Let the induction hypothesis be $p(n) =$ "$2^n - 1 \leq 3^n$".

Base case: Our base case is for $n = 0$. We have $2^0 - 1 = 0 \leq 3^0$.

Inductive case: Assume $p(n)$ is true. Then $2^n - 1 \leq 3^n$. Looking at $p(n+1)$:

$$\begin{aligned}
2^{n+1} - 1 &= 2 \cdot 2^n - 2 + 1 && \text{manipulating} \\
&= 2(2^n - 1) + 1 && \text{manipulating} \\
&\leq 2 \cdot 3^n + 1 && \text{by } p(n) \\
&\leq 2 \cdot 3^n + 3^n && \text{since } 1 \leq 3^n \text{ for } n > 0 \\
&= 3^n(2+1) && \text{manipulating} \\
&\leq 3^{n+1}
\end{aligned}$$

Therefore, by induction $2^n - 1 \leq 3^n$ for $n \geq 0$. □

Example 5:
Prove $3|n^3 - n$, $\forall n \geq 0$.

Solution:
Proof. We use a proof by induction on an integer n. Let the induction hypothesis be: $p(n) =$ "$3 \mid n^3 - n$".

Base case: Our base case is for $n = 0$, for which we have that 3 divides $0^3 - 0 = 0$.

Inductive case: Assume $p(n)$ is true. Then 3 divides $n^3 - n$. This means $n^3 - n = 3k$ for some $k \in \mathbb{Z}$. Looking at $p(n+1)$:

$$\begin{aligned}
(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 && \text{expanding} \\
&= n^3 - n + 3n^2 + 3n && \text{manipulating} \\
&= 3k + 3(n^2 + n) && \text{by } p(n) \\
&= 3k + 3m && \text{factoring, and } n^2 + n \in \mathbb{Z} \\
&= 3(k + m) && k + m \text{ is also an integer}
\end{aligned}$$

Therefore, by induction 3 divides $n^3 - n$. □

Example 6:

Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ using induction.

Solution:

Proof. We use a proof by induction on $n \in \mathbb{N}$. Let the induction hypothesis be $p(n) =$ "$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$" on the induction variable n.

Base case: Our base case is for $n = 1$. We have

$$1 = \frac{1(1+1)}{2} = 1.$$

Inductive case: Assume $p(n)$ is true. Then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Looking at the sum of the first $n+1$ integers yields:

$$\begin{aligned}
1 + 2 + \cdots + n + (n+1) &= (1 + 2 + \cdots + n) + (n+1) && \text{breaking out the } n+1 \\
&= \frac{n(n+1)}{2} + (n+1) && \text{by } p(n) \\
&= \frac{n(n+1) + 2(n+1)}{2} && \text{manipulating} \\
&= \frac{(n+1)(n+2)}{2} && \text{manipulating} \\
&= p(n+1)
\end{aligned}$$

Therefore $p(n+1)$ is true. By induction, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

□