The Stamp Problem.

Suppose you have unlimited 5¢ and 8¢ stamps.

- Can you make 4¢?
- Can you make 28¢?
- Can you make 85,694¢?

Goals for today:

one of the most important topics in our course.

- state the basic steps of a proof by induction,
- prove some simple propositions using induction.

If you have 30¢ in stamps (6 x 5¢), can you make 31¢, 32¢, 33¢, 34¢?

only 5¢
8¢ stamps.

$$5¢\ 5¢\ 5¢\ \cancel{5¢}\ \cancel{5¢}\ \cancel{5¢} + \boxed{8} + \boxed{8} = 31¢$$

$$\cancel{5¢}\ \cancel{5¢}\ \cancel{5¢}\ \cancel{5¢}\ \cancel{5¢}\ \cancel{5¢} + 4 \times \boxed{8} = 32¢$$

$$5¢\ 5¢\ 5¢\ 5¢\ 5¢\ \cancel{5¢} + \boxed{8} = 33¢$$

$$5¢\ 5¢\ \cancel{5¢}\ \cancel{5¢}\ \cancel{5¢}\ \cancel{5¢} + 3 \times \boxed{8} = 34¢$$

main idea: make higher values from lower values.

The Principle of Mathematical Induction

main idea: show that some predicate is true for $\underbrace{(n+1)}_{\text{some variable}}$ assuming it's true for some value n.

1. **State your method:** We use a proof by induction.
2. **Identify your induction variable:** introduce n (stamp value)
3. **State your induction hypothesis:**
   Let the I.H. be the predicate p(n) = "we can make a value of n¢ from 5¢ and 8¢ stamps"
4. **Prove your base case:**
   What is the lowest value of n?
5. **Prove the inductive case:**
   Prove p(n+1) is true assuming p(n) true.
6. **Conclude:**
   Therefore, by induction on n, p(n) true for all n ≥ (base value).

An idea about stamps:

If $\underbrace{\text{we have} \geq 28¢}_{p}$, then $\underbrace{\text{we either have} \geq 3 \text{ x } 5¢ \text{ or} \geq 3 \text{ x } 8¢}_{q}$.

$\neg q \to \neg p$

$\neg p$: we have $< 28¢$

$\neg q$: $\neg$ (have ≥ 3 $\boxed{5}$ or ≥ 3 $\boxed{8}$)

have < 3 $\boxed{5}$ AND < 3 $\boxed{8}$

at most ~~most~~ ~~most~~ 2 $\boxed{5}$ AND 2 $\boxed{8}$, so we have $26¢ < 28¢$

why??

(1) $n - 3\,\boxed{5} + 2\,\boxed{8} = n+1$

(2) $n - 3\,\boxed{8} + 5\,\boxed{5} = n+1$

Prove we can make a stamp value $\geq$ 28¢ using only 5¢ and 8¢ stamps.

proof: We use a proof by induction on the stamp value n.

Let the I.H. be the predicate:

$p(n) =$ "we can create a stamp value of n¢ using 5¢ and 8¢ stamps."

We will prove $p(n)$ is true for all $n \geq 28$¢.

base case: For $n = 28$¢, we can use 4×5¢ $+ 1 \times 8$¢ stamps.

inductive step: Assume $p(n)$ true, i.e. we can create a value of n¢. We need to show $p(n+1)$ true, i.e. show we can create a value of $(n+1)$¢. By lemma (previous slide) we either have 3×5¢ or 3×8¢. We can then create $(n+1)$¢ by either subtracting 3×5¢ and adding 2×8¢ or subtracting 3×8¢ and adding 5×5¢. This means $p(n+1)$ true.

Therefore, by induction on n, $p(n)$ is true. $\square$

A numbery example: prove $3 \mid (n^3 - n),\ \forall n \geq 0.$

$n^3 - n = 3k \quad k \in \mathbb{Z}$

proof: We use a proof by induction on n.

Let the IH be the predicate:

$p(n) =$ " $3 \mid n^3 - n$ "

We will prove $p(n)$ true for all $n \geq 0$.

base case: For $n = 0$, $3 \mid 0$ since $0 \cdot 3 = 0$.

inductive step: Assume $p(n)$ true, that is $3 \mid n^3 - n$, so $n^3 - n = 3k$, $k \in \mathbb{Z}$.

We will show $p(n+1)$ true. In other words show

$(n+1)^3 - (n+1)$ is some multiple of 3.

Starting with $(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + \cancel{1} - (n + \cancel{1})$

$= n^3 - n + 3(n^2 + n)$

$= 3k + 3(n^2 + n) = 3m$ $\quad k + n^2 + n \in \mathbb{Z}$.

meaning $p(n+1)$ true.

Therefore, by induction $p(n)$ true. $\square$