

The Stamp Problem.

Suppose you have unlimited 5¢ and 8¢ stamps.

- Can you make 4¢?
- Can you make 28¢?
- Can you make 85,694¢?

Goals for today:

- state the basic steps of a proof by induction,
- prove some simple propositions using induction.

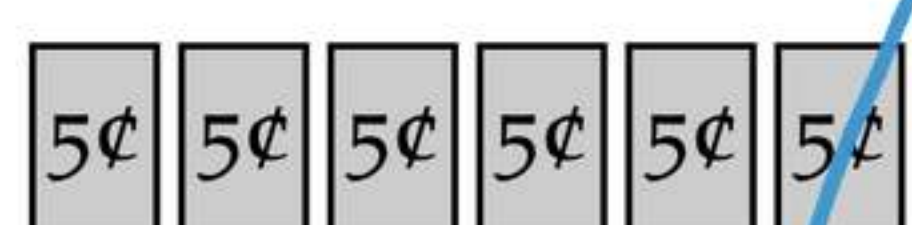
→ one of the most important topics in our course.

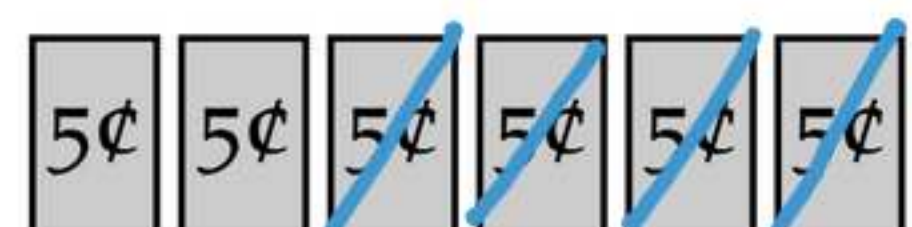

If you have 30¢ in stamps (6 x 5¢), can you make 31¢, 32¢, 33¢, 34¢?

 +  +  = 31¢

only 5[¢]
8[¢] stamps.

 + 4 x  = 32¢

 +  = 33¢

 + 3 x  = 34¢

main idea: make higher values from lower values.

The Principle of Mathematical Induction

main idea: show that some predicate is true for $(n+1)$ assuming it's true for some value n . ^{some variable}

1. State your method: We use a proof by induction.

2. Identify your induction variable: introduce n (stamp value)

3. State your induction hypothesis:

Let the I.H. be the predicate $p(n) =$ "we can make a value of $n^{\$}$ from $5^{\$}$ and $8^{\$}$ stamps"

4. Prove your base case:

What is the lowest value of n ?

5. Prove the inductive case:

Prove $p(n+1)$ is true assuming $p(n)$ true.

6. Conclude:

Therefore, by induction on n , $p(n)$ true for all $n \geq$ (base value).

An idea about stamps:

If we have $\geq 28\text{¢}$, then we either have $\geq 3 \times 5\text{¢}$ or $\geq 3 \times 8\text{¢}$.

p

q

$\neg q \rightarrow \neg p$

$\neg p$: we have $< 28\text{¢}$

$\neg q$: \neg (have ≥ 3 $\boxed{5}$ or ≥ 3 $\boxed{8}$)
have < 3 $\boxed{5}$ AND < 3 $\boxed{8}$

at most most most 2 $\boxed{5}$ AND 2 $\boxed{8}$, so we have $26\text{¢} < 28\text{¢}$

why??

$$\textcircled{1} n - 3\boxed{5} + 2\boxed{8} = n+1$$

$$\textcircled{2} n - 3\boxed{8} + 5\boxed{5} = n+1$$

Prove we can make a stamp value $\geq 28¢$ using only 5¢ and 8¢ stamps.

proof: We use a proof by induction on the stamp value n .

Let the I.H. be the predicate:

$p(n)$ = "we can create a stamp value of $n¢$ using 5¢ and 8¢ stamps"

We will prove $p(n)$ is true for all $n \geq 28¢$.

base case: For $n = 28¢$, we can use $4 \times 5¢ + 1 \times 8¢$ stamps.

inductive step: Assume $p(n)$ true, i.e. we can create a value of $n¢$. We need to show $p(n+1)$ true, i.e. show we can create a value of $(n+1)¢$. By lemma (previous slide) we either have $3 \times 5¢$ or $3 \times 8¢$. We can then create $(n+1)¢$ by either subtracting $3 \times 5¢$ and adding $2 \times 8¢$ or subtracting $3 \times 8¢$ and adding $5 \times 5¢$. This means $p(n+1)$ true.

Therefore, by induction on n , $p(n)$ is true. □

A number example: prove $3 \mid (n^3 - n), \forall n \geq 0$.

$$n^3 - n = 3k \quad k \in \mathbb{Z}$$

proof: We use a proof by induction on n .

Let the IH be the predicate:

$$p(n) = '3 \mid n^3 - n'$$

We will prove $p(n)$ true for all $n \geq 0$.

base case: For $n = 0$, $3 \mid 0$ since $0 \cdot 3 = 0$.

inductive step: Assume $p(n)$ true, that is $3 \mid n^3 - n$, so $n^3 - n = 3k, k \in \mathbb{Z}$.
We will show $p(n+1)$ true. In other words, show

$(n+1)^3 - (n+1)$ is some multiple of 3.

$$\begin{aligned} \text{Starting with } (n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - (n+1) \in \mathbb{Z}. \\ &= n^3 - n + 3(n^2 + n) \rightarrow k + 3n^2 + 3n \\ &= 3k + 3(n^2 + n) = 3m \end{aligned}$$

Therefore, by induction $p(n)$ true. \square