The Stamp Problem.

Suppose you have unlimited 5¢ and 8¢ stamps.

- Can you make 4¢?
- Can you make 28¢?
- Can you make 85,694¢?

Goals for today:

- state the basic steps of a proof by induction,
- prove some simple propositions using induction.
If you have 30¢ in stamps (6 x 5¢), can you make 31¢, 32¢, 33¢, 34¢?

- \[ 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad + \quad \boxed{8} \quad + \quad \boxed{8} \quad = \quad 31¢ \]
- \[ 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad + \quad 4 \times \boxed{8} \quad = \quad 32¢ \]
- \[ 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad + \quad \boxed{8} \quad = \quad 33¢ \]
- \[ 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad 5¢ \quad + \quad 3 \times \boxed{8} \quad = \quad 34¢ \]

**main idea:** Make higher values from lower values.
The Principle of Mathematical Induction

main idea: show that some predicate is true for \((n+1)\) assuming it’s true for some value \(n\).

1. State your method: We use a proof by induction.

2. Identify your induction variable: introduce \(n\) (stamp value)

3. State your induction hypothesis:
   Let the I.H. be the predicate \(p(n)\) = "we can make a value of \(n\) from 5\(^{st}\) and 8\(^{th}\) stamps"

4. Prove your base case:
   What is the lowest value of \(n\)?

5. Prove the inductive case:
   Prove \(p(n+1)\) is true assuming \(p(n)\) true.

6. Conclude:
   Therefore, by induction on \(n\), \(p(n)\) true for all \(n \geq \text{(base value)}\).
An idea about stamps:
If we have \( \geq 28\text{¢} \), then we either have \( \geq 3 \times 5\text{¢} \) or \( \geq 3 \times 8\text{¢} \).

\[ p \land q \]
\[ -p : \text{we have } \leq 28\text{¢} \]
\[ -q : - ( \text{have } \geq 3 \text{ 5¢ or } \geq 3 \text{ 8¢}) \]
\[ \text{have } < 3 \text{ 5¢ AND } < 3 \text{ 8¢} \]

at most at most at most \( 2 \text{ 5¢ AND 2 8¢} \), so we have \( 26\text{¢} < 28\text{¢} \)

\[ n - 3 \text{ 5¢} + 2 \text{ 8¢} = n+1 \]
\[ n - 3 \text{ 8¢} + 5 \text{ 5¢} = n+1 \]
Prove we can make a stamp value $\geq 28\$ using only 5\$ and 8\$ stamps.

**proof:** We use a proof by induction on the stamp value $n$.

Let the I.H. be the predicate:

$$p(n) = \text{"we can create a stamp value of n\$ using 5\$ and 8\$ stamps."}$$

We will prove $p(n)$ is true for all $n \geq 28\$.

**base case:** For $n = 28\$, we can use $4 \times 5\$ + $1 \times 8\$ stamps.

**inductive step:** Assume $p(n)$ true, i.e. we can create a value of $n\$. We need to show $p(n+1)$ true, i.e. show we can create a value of $(n+1)\$. By lemma (previous slide) we either have $3 \times 5\$ or $3 \times 8\$. We can then create $(n+1)\$ by either subtracting $3 \times 5\$ and adding $2 \times 8\$ or subtracting $3 \times 8\$ and adding $5 \times 5\$. This means $p(n+1)$ true.

Therefore, by induction on $n$, $p(n)$ is true. \qed
A numbery example: prove $3 \mid (n^3 - n)$, $\forall n \geq 0$.

\[ n^3 - n = 3k \quad k \in \mathbb{Z} \]

**Proof:** We use a proof by induction on $n$.

Let the IH be the predicate:

\[ p(n) = 3 \mid n^3 - n \quad \]

We will prove $p(n)$ true for all $n \geq 0$.

**Base case:** For $n = 0$, $3 \mid 0$ since $0 \cdot 3 = 0$.

**Inductive step:** Assume $p(n)$ true, that is $3 \mid n^3 - n$, so $n^3 - n = 3k$, $k \in \mathbb{Z}$.

We will show $p(n+1)$ true. In other words, show

\[(n+1)^3 - (n+1) \text{ is some multiple of } 3.\]

Starting with $(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - (n+1) \in \mathbb{Z}$.

\[ = n^3 - n + 3(n^2 + n) \quad k + h^2 \quad \text{is some multiple of } 3. \]

Therefore, by induction $p(n)$ true.