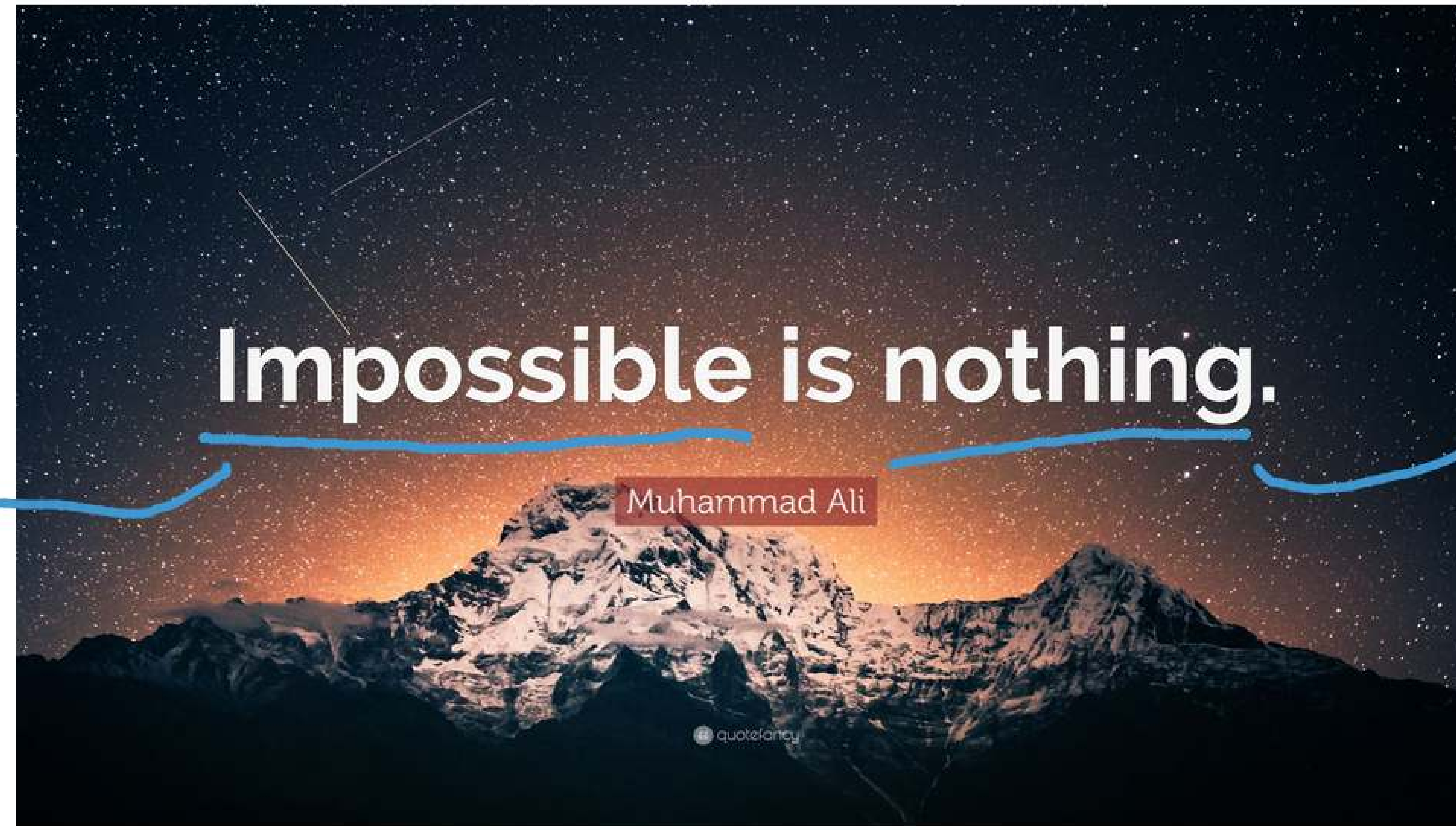
How can we prove this?

"everything is possible"



p

$\neg p$ "possible"

q

$\neg q$ "everything"

Goals for today:

- prove an implication using the contrapositive,
- prove by assuming a contradiction,
- prove an if-and-only-if.

But first: proving with examples.

Which could be proved using an $\underbrace{\text{example}}$?

or maybe a counterexample?

1. $\forall x \in S, p(x)$ X

2. $\forall x \in S, \neg p(x)$ X

3. $\neg \exists x \in S \colon p(x)$ $\forall x \in S, \neg p(x)$ X

4. $\neg \forall x \in S, p(x)$ $\exists x \in S : \neg p(x)$ ✓

Proof Method #3: contrapositive.

remember rule #3 $\dfrac{p \to q}{\therefore \neg q \to \neg p}$

Example: $\underbrace{\textit{If } a^2 \textit{ is not divisible by } 4}_{p}$, *then* $\underbrace{a \textit{ is odd}}_{q}$.

$\neg p$ a^2 is divisible by 4

$\neg q$ a is even.

proof: We prove the contrapositive.

Let $a \in \mathbb{Z}$ and suppose a is even,

which means $a = 2k$ for some $k \in \mathbb{Z}$.

This means $a^2 = (2k)^2 = 4k^2 = 4m$ for some $m \in \mathbb{Z}$,

which means a^2 is divisible by 4.

Therefore, if a^2 is not divisible by 4, then a is odd.

$\square$

Prove: *If $\underbrace{a^2 \text{ is even}}_{P}$, then $\underbrace{a \text{ is even}}$.*

<u>Proof:</u> We prove the contrapositive.

Let $a \in \mathbb{Z}$ be an odd integer, meaning $a = 2k+1$ for some $k \in \mathbb{Z}$.

Then $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

$= 2m + 1$ for $m \in \mathbb{Z}$.

(which means a^2 is odd).

Therefore, if a^2 is even, then a is even.

$\square$

Proof Method #4: contradiction.

- Assume $\neg p$
- try to show a contradiction happens
- therefore, p must be true.

Example: *Prove that* $\underbrace{\sqrt{2} \text{ is irrational.}}_{p}$

$\frac{2}{3}$ $\frac{4}{6}$

<u>proof:</u> We use a proof by contradiction.

Suppose $\sqrt{2}$ is rational. This means $\exists m,n \in \mathbb{Z} : \frac{m}{n} = \sqrt{2}$.

Without loss of generality assume $\frac{m}{n}$ is simplified to lowest terms. (by lemma on last slide)

Then $m^2 = 2n^2$, meaning m^2 is even and m is also even.

This means $m = 2k$ for some $k \in \mathbb{Z}$. Then $(2k)^2 = 2n^2$, so

$2 \cdot \cancel{4}k^2 = \cancel{2}n^2$ which means n^2 is even, so n is also even. This

means $n = 2a$ for $a \in \mathbb{Z}$. Then $\sqrt{2} = \frac{\boxed{2}k}{\boxed{2}a}$ contradiction, which is a contradiction

since we said $\frac{m}{n}$ was simplified.

Therefore, $\sqrt{2}$ is irrational. $\square$

[Exercise 1]

Suppose $a \in \mathbb{Z}$. Prove that $\overbrace{14 \mid a}^{p}$ *if and only if* $\overbrace{7 \mid a \text{ and } 2 \mid a}^{q}$.

$\frac{a}{14}$ is an integer

iff proofs:

$p \to q \wedge q \to p$
(1) (2)

proof. SKETCH

(1) assume $14 \mid a$, so $a = 14k$ $k \in \mathbb{Z}$.

$$a = (2)\underbrace{(7)k}_{m} = 2m = 7n$$

(2) assume $7 \mid a$ and $2 \mid a$, show $14 \mid a$

$a = 7k$ $k \in \mathbb{Z}$ $\rightarrow$ $a = 7k$ is even so k is even, $k = 2n$ $n \in \mathbb{Z}$

$a = 2m$ $m \in \mathbb{Z}$ $\rightarrow$ a is even.

$a = 7(2n) = 14n$.