Learning objectives:
- identify errors in inductive proofs
- prove correctness of recursive programs with induction

Last time, we introduced induction. Let’s warm up by trying to identify errors in the following proof.

Example 1:
The following sentences are used to prove the following proposition. Put them in order, and correct any errors.

Prove that $7^n - 1$ is a multiple of 6 for all $n \geq 0$.

- Then there exists an integer $b$ such that $7^k - 1 = 6b$.
- Because $b$ is an integer, $7b + 1$ is an integer, so $p(k + 1)$ is true.

Inductive step: Let $k \geq 1$ and assume that $p(k)$ is true.

- Let the induction hypothesis be the predicate: $p(n) = 7^n - 1$ is a multiple of 6 for all $n \geq 0$.
- Base case: $p(1)$ is true because $7^1 - 1 = 6$, which is a multiple of 6 since $6 \times 1 = 6$.
- We use a proof by induction.
- Let the induction hypothesis $p(n)$ is true.
- Therefore, by induction on $n$, $p(n)$ is true for all $n \geq 0$.
- Multiplying both sides by 7, we get $7^{k+1} - 1 = 6(7b + 1)$.

Solution:
Proof. We use a proof by induction. Let the induction hypothesis be the predicate: $p(n) = 7^n - 1$ is a multiple of 6. We will prove that $p(n)$ is true for all $n \geq 0$.

- Base case: $p(0)$ is true because $7^0 - 1 = 0$, which is a multiple of 6 since $6 \times 0 = 0$.

- Inductive step: Let $n \geq 0$ and assume that $p(n)$ is true. Then there exists an integer $b$ such that $7^n - 1 = 6b$. Multiplying both sides by 7 and adding 6 to both sides, we get $7^{n+1} - 1 = 6(7b + 1)$. Since $b$ is an integer, then $7b + 1$ is also an integer, so $p(n + 1)$ is true.

Therefore, by induction on $n$, $p(n)$ is true for all $n \geq 0$.

Note that in the second sentence, it is incorrect to keep the for all quantifier, because $p(n)$ would no longer be a predicate in that case (it still needs to depend on the input variable $n$).
1 Induction with sets

We’ve done a bunch of number-y examples, so let’s do one with sets. This is good practice for the types of proofs we will do later with graphs.

Define the power set as the set of all possible subsets of a set. For example, for a set \( A = \{a, b, c\} \), the power set is
\[
P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.
\]

Example 2:
Prove that the cardinality of the power set with \( n \) elements is \(|\mathcal{P}(A)| = 2^n|\).

Solution:
Proof. We use a proof by induction. Let the induction hypothesis be: \( p(n) = \text{the cardinality of the power set with } n \text{ elements is } |\mathcal{P}(A)| = 2^n|\).

Base case: Our base case is for \( n = 0 \), in which we have the single emptyset. Therefore, \(|\mathcal{P}(A)| = 2^0 = 1|\).

Inductive case: Assume \( p(n) \) is true. Then the cardinality of the power set of a set with \( n \) elements is \( 2^n \). Now, consider the set \( A_{n+1} \) with \( n + 1 \) elements: \( \{a_1, a_2, \ldots, a_n, a_{n+1}\} \). We want to show that \( \mathcal{P}(A_{n+1}) = 2^{n+1} \). Remove the last element of \( A_{n+1} \), to create a set with \( n \) elements: \( A_n = \{a_1, a_2, \ldots, a_n\} \).

By the definition of the power set, \( \mathcal{P}(A_{n+1}) \) includes every element in \( \mathcal{P}(A_n) \) paired with \( a_{n+1} \), along with every element in \( \mathcal{P}(A_n) \):
\[
\mathcal{P}(A_{n+1}) = \mathcal{P}(A_n) \cup \{x \cup a_{n+1} \mid x \in \mathcal{P}(A_n)\}.
\]

The cardinality of \(|\mathcal{P}(A_{n+1})|\) is the sum of the cardinalities of both sets, minus the cardinality of their intersection. Therefore,
\[
|\mathcal{P}(A_{n+1})| = |\mathcal{P}(A_n)| + |\{x \cup a_{n+1} \mid x \in \mathcal{P}(A_n)\}| + |\{x \cup a_{n+1} \mid x \in \mathcal{P}(A_n)\}|
= 2^n + |\{x \cup a_{n+1} \mid x \in \mathcal{P}(A_n)\}|
= 2^n + 2^n
= 2 \cdot 2^n
= 2^{n+1}
\]

Therefore, by induction the cardinality of the power set of a set with \( n \) elements is \( 2^n \). \( \square \)
Mathematical induction has a lot of similarities with recursion. Remember, that when writing recursive programs, it is very important to make sure you have a base case and recursive case, similar to the base case and inductive steps used in a proof by induction. It is important to make sure your recursive programs work correctly, so we will now practice proving the correctness of a few recursive functions.

Consider the following pseudocode which describes a recursive solution for reversing a string.

```
reverseString(s)
    input: s (string)
    output: reversed string
    1 if length(s) == 1 # base case
        return s
    2 else # recursive case
        return reverseString( s[1:] ) + s[0]
```

Example 3:
Prove that the reverseString function listed in Algorithm 1 is correct.

**Solution:**
Proof. We use a proof by induction. Let \( p(n) \) be the predicate that reverseString correctly reverses an input string of length \( n \). We will prove that reverseString correctly reverses strings for \( n \geq 1 \).

Base case: Consider strings of length \( n = 1 \). The reverse of this string is just the string itself, which Line 2 correctly returns.

Inductive case: Let \( n > 1 \) and assume that \( p(n) \) is true. That is, reverseString correctly reverses strings of length \( n \). Now consider a string of length \( n + 1 \). Since \( n \geq 1 \), the algorithm jumps to the recursive step on Line 4. Remove the first character from this string to create a string of length \( n \) and pass this into reverseString. By \( p(n) \), then this string of length \( n \) is correctly reversed and we need only move the first character (which we removed to create a string of length \( n \)) to the end. This is what Line 4 does, so \( p(n + 1) \) is true.

Therefore, by induction on the length of the input strings \( n \), reverseString works correctly.

In the last example, we proved the correctness of the stringReverse
function. Sometimes, we want to prove our recursive function achieves some property.

**Example 4:**
Prove that the total length drawn by Algorithm 2 is \( L \frac{1 - a^n}{1 - a} \), when called with \( n \) generations \( n > 0 \) and a factor \( 0 < a < 1 \).

**Solution:**
*Proof.* We use a proof by induction on the number of generations \( n \). Let the induction hypothesis be \( p(n) = \text{Algorithm 2 draws a total length of } L \frac{1 - a^n}{1 - a} \).

**Base case:** Our base case is at \( n = 0 \), in which case nothing is drawn. Line 2 correctly draws nothing at \( n = 0 \), which agrees with \( p(0) = L \frac{1 - a^0}{1 - a} = 0 \) since \( a \neq 0 \).

**Inductive step:** Assume \( p(n) \) is true. That is, the total length drawn by \( \text{spiral}(n, L, a) = L \frac{1 - a^n}{1 - a} \). Now consider the call \( \text{spiral}(n + 1, L, a) \) for \( n > 0 \). We are now in the recursive case. Since \( p(n) \) is true, then Line 6 draws a spiral of length \( aL \frac{1 - a^n}{1 - a} \). Don’t forget the \( a \) in front because the \( L \) that is passed to \( \text{spiral} \) is \( aL \). The only additional drawing happens on Line 5, which incurs an additional length of \( L \). Therefore, the total length drawn at \( n + 1 \) is

\[
aL \frac{1 - a^n}{1 - a} + L = L \frac{a - a^{n+1} + 1 - a}{1 - a} = L \frac{1 - a^{n+1}}{1 - a}.
\]

Thus, by induction on the number of generations \( n \), \( p(n) \) is true.

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**Algorithm 2:** Recursive function for drawing a spiral. Assume that the function \( \text{draw_line} \) draws a straight line of some input length \( L \) and \( \text{turn_left} \) turns the heading by some input angle (in degrees). This is very similar to the \( \text{forward} \) and \( \text{left} \) functions in Python’s Turtle graphics module.

```python
spiral(n, L, a)

input: n: number of generations, L: length to draw, a: factor to decrease length in next generation
output: none
1 if n == 0  # base case
2    return
3 else  # recursive case
4    turn_left(30 degrees)  # turns left by 30 degrees
5    draw_line(L)  # draws a straight line of length L
6    spiral(n - 1, aL, a)
```

Sample output of Algorithm 2 for \( n = 50, L = 50 \) and \( a = 0.95 \).