Learning objectives:

- ☐ identify errors in inductive proofs
- ☐ prove correctness of recursive programs with induction

Last time, we introduced induction. Let's warm up by trying to identify errors in the following proof.

Be careful!

Be careful with your implications! It is incorrect to show that $p(k+1) \implies p(k)$.

Example 1:
The following sentences are used to prove the following proposition. Put them in order, and correct any errors.

Prove that $7^n - 1$ is a multiple of 6 for all $n \geq 0$.

- Then there exists an integer b such that $7^k - 1 = 6b$.
- Because b is an integer, $7b + 1$ is an integer, so $p(k+1)$ is true.
- **Inductive step:** Let $k \geq 1$ and assume that $p(k)$ is true.
- Let the induction hypothesis be the predicate: $p(n) = 7^n - 1$ *is a multiple of* 6 *for all* $n \geq 0$.
- **Base case:** $p(1)$ is true because $7^1 - 1 = 6$, which is a multiple of 6 since $6 \times 1 = 6$.
- We use a proof by induction.
- Let the induction hypothesis $p(n)$ is true.
- Therefore, by induction on n, $p(n)$ is true for all $n \geq 0$.
- Multiplying both sides by 7, we get $7^{k+1} - 1 = 6(7b+1)$.

Solution:
Proof. We use a proof by induction. Let the induction hypothesis be the predicate: $p(n) = 7^n - 1$ *is a multiple of* 6. We will prove that $p(n)$ is true for all $n \geq 0$.

- **Base case:** $p(0)$ is true because $7^0 - 1 = 0$, which is a multiple of 6 since $6 \times 0 = 0$.
- **Inductive step:** Let $n \geq 0$ and assume that $p(n)$ is true. Then there exists an integer b such that $7^n - 1 = 6b$. Multiplying both sides by 7 and adding 6 to both sides, we get $7^{n+1} - 1 = 6(7b+1)$. Since b is an integer, then $7b + 1$ is also an integer, so $p(n+1)$ is true.

Therefore, by induction on n, $p(n)$ is true for all $n \geq 0$. ☐

Note that in the second sentence, it is incorrect to keep the *for all* quantifier, because $p(n)$ would no longer be a predicate in that case (it still needs to depend on the input variable n).

1 Induction with sets

We've done a bunch of number-y examples, so let's do one with sets. This is good practice for the types of proofs we will do later with graphs.

Define the *power set* as the set of all possible subsets of a set. For example, for a set $A = \{a, b, c\}$, the power set is

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Example 2:
Prove that the cardinality of the power set with n elements is $|\mathcal{P}(A)| = 2^n$.

Solution:
Proof. We use a proof by induction. Let the induction hypothesis be: $p(n)$ = *the cardinality of the power set with n elements is* $|\mathcal{P}(A)| = 2^n$.

Base case: Our base case is for $n = 0$, in which we have the single emptyset. Therefore, $|\mathcal{P}(A)| = 2^0 = 1$.

Inductive case: Assume $p(n)$ is true. Then the cardinality of the power set of a set with n elements is 2^n. Now, consider the set A_{n+1} with $n+1$ elements: $\{a_1, a_2, \ldots, a_n, a_{n+1}\}$. We want to show that $\mathcal{P}(A_{n+1}) = 2^{n+1}$. Remove the last element of A_{n+1}, to create a set with n elements: $A_n = \{a_1, a_2, \ldots, a_n\}$. By the definition of the power set, $\mathcal{P}(A_{n+1})$ includes every element in $\mathcal{P}(A_n)$ paired with a_{n+1}, along with every element in $\mathcal{P}(A_n)$:

$$\mathcal{P}(A_{n+1}) = \mathcal{P}(A_n) \cup \{x \cup a_{n+1} \mid x \in \mathcal{P}(A_n)\}.$$

The cardinality of $|\mathcal{P}(A_{n+1})|$ is the sum of the cardinalities of both sets, minus the cardinality of their intersection. Therefore,

$$\begin{aligned}
|\mathcal{P}(A_{n+1})| &= |\mathcal{P}(A_n)| + |\{x \cup a_{n+1} \mid x \in \mathcal{P}(A_n)\}| \\
&= 2^n + |\{x \cup a_{n+1} \mid x \in \mathcal{P}(A_n)\}| && \text{by } p(n) \\
&= 2^n + 2^n && 2^n \text{ elements of } \mathcal{P}(A_n) \text{ are paired with } a_{n+1} \\
&= 2 \cdot 2^n \\
&= 2^{n+1}
\end{aligned}$$

Therefore, by induction the cardinality of the power set of a set with n elements is 2^n. □

Mathematical induction has a lot of similarities with *recursion*. Remember, that when writing recursive programs, it is very important to make sure you have a **base case** and **recursive case**, similar to the base case and inductive steps used in a proof by induction. It is important to make sure your recursive programs work correctly, so we will now practice proving the correctness of a few recursive functions.

Consider the following pseudocode which describes a recursive solution for reversing a string.

```
reverseString(s)

    input: s (string)
    output: reversed string
1   if length(s) == 1      # base case
2       return s
3   else    # recursive case
4       return reverseString( s[1:] ) + s[0]
```

Algorithm 1: Recursive function for reversing a string. Here, the string indexing starts at 0 (like `Python` or `C`-like languages). Elements (characters) of the string can be accessed with square brackets ([]), and a substring can be extracted with a colon (*start* : *end*).

Pseudocode?

This is often useful when you want to describe a sequence of steps as you would in a programming language without restricting yourself to specific language. You can use basic keywords like **if**, **else**, **for**, **return** and also highlight when you might be calling a separate function. The focus of pseudocode is truly on the *algorithm* along with the corresponding inputs and outputs, not on the semantics of your code.

Example 3:
Prove that the **reverseString** function listed in Algorithm 1 is correct.

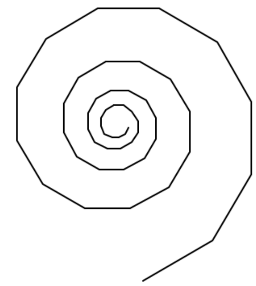
Solution:
Proof. We use a proof by induction. Let $p(n)$ be the predicate that **reverseString** correctly reverses an input string of length n. We will prove that **reverseString** correctly reverses strings for $n \geq 1$.

Base case: Consider strings of length $n = 1$. The reverse of this string is just the string itself, which Line 2 correctly returns.

Inductive case: Let $n > 1$ and assume that $p(n)$ is true. That is, **reverseString** correctly reverses strings of length n. Now consider a string of length $n+1$. Since $n \geq 1$, the algorithm jumps to the recursive step on Line 4. Remove the first character from this string to create a string of length n and pass this into **reverseString**. By $p(n)$, then this string of length n is correctly reversed and we need only move the first character (which we removed to create a string of length n) to the end. This is what Line 4 does, so $p(n+1)$ is true.

Therefore, by induction on the length of the input strings n, **reverseString** works correctly. □

In the last example, we proved the correctness of the **stringReverse** function. Sometimes, we want to prove our recursive function achieves some property.



Sample output of Algorithm 2 for $n = 50$, $L = 50$ and $\alpha = 0.95$.

Example 4:

Prove that the total length drawn by Algorithm 2 is $L\frac{1-\alpha^n}{1-\alpha}$, when called with n generations $n > 0$ and a factor $0 < \alpha < 1$.

Solution:

Proof. We use a proof by induction on the number of generations n. Let the induction hypothesis be $p(n) =$ *Algorithm 2 draws a total length of* $L\frac{1-\alpha^n}{1-\alpha}$.

Base case: Our base case is at $n = 0$, in which case nothing is drawn. Line 2 correctly draws nothing at $n = 0$, which agrees with $p(0) = L\frac{1-\alpha^0}{1-\alpha} = \frac{0}{1-\alpha} = 0$ since $\alpha \neq 0$.

Inductive step: Assume $p(n)$ is true. That is, the total length drawn by **spiral**$(n, L, \alpha) = L\frac{1-\alpha^n}{1-\alpha}$. Now consider the call **spiral**$(n+1, L, \alpha)$ for $n > 0$. We are now in the recursive case. Since $p(n)$ is true, then Line 6 draws a spiral of length $\alpha L\frac{1-\alpha^n}{1-\alpha}$. Don't forget the α in front because the L that is passed to **spiral** is αL. The only additional drawing happens on Line 5, which incurs an additional length of L. Therefore, the total length drawn at $n+1$ is

$$\alpha L\frac{1-\alpha^n}{1-\alpha} + L = L\frac{\alpha - \alpha^{n+1} + 1 - \alpha}{1-\alpha} = L\frac{1-\alpha^{n+1}}{1-\alpha}.$$

Thus, by induction on the number of generations n, $p(n)$ is true. □

```
spiral(n, L, α)

   input: n: number of generations, L: length to draw,
   α: factor to decrease length in next generation
   output: none
1  if n == 0    # base case
2      return
3  else  # recursive case
4      turn_left(30 degrees)    # turns left by 30 degrees
5      draw_line(L)    # draws a straight line of length L
6      spiral(n − 1, αL, α)
```

Algorithm 2: Recursive function for drawing a spiral. Assume that the function **draw_line** draws a straight line of some input length L and **turn_left** turns the heading by some input angle (in degrees). This is very similar to the **forward** and **left** functions in Python's Turtle graphics module.