From last class: equivalent ways of "saying" $p \implies q$.

IMPLICATIONS

We will discuss several other important ways in which propositions can be combined.

> Let p and q be propositions. The *implication* $p \to q$ is the proposition that is false when p is true and q is false, and true otherwise. In this implication p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

The truth table for the implication $p \to q$ is shown in Table 5. An implication is sometimes called a **conditional statement.**

Because implications play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \to q$. You will encounter most if not all of the following ways to express this implication:

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"

The implication $p \to q$ is false only in the case that p is true, but q is false. It is true when both p and q are true, and when p is false (no matter what truth value q has).

A useful way to understand the truth value of an implication is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is:

$\overbrace{\text{go swimming}}^{w}$ only if $\underbrace{\text{it's sunny}}_{s}$

$w \to s$

go swimming (w)	it's sunny (s)	valid?
T	T	T
T	F	F
F	T	T
F	F	T

same truth table as $w \to s$

General structure of a proof.

→ omit this only if it's a <u>direct proof</u> with a <u>single</u> case.

- State your plan. We use a proof by [insert proof method].
- Introduce your variables. Let n be an integer.
  Let $x \in \mathbb{R}$.
- State your assumptions.
  Suppose [insert statement]. Assume [statement] ...
- Write your proof as an essay (not a calculation).
  use complete sentences.
- Revise and simplify.
  ↳ can you remove steps without sacrificing clarity?
  → use scrap paper!
- Finish.
  Therefore, ...

  □ always!

Summary of what we can do so far:

- Use mathematical notation to represent things. → propositions, predicates, sets, logical ops, quantifiers
- Use a truth table to check a conclusion.
- Derive new conclusions using our 4 rules of inference.

What do we want to do today?

- Prove conclusions (propositions, theorems, lemmas).
- Divide these proofs into cases (if necessary).
- Use correct proof structure.

Method #1: Massage

left-hand-side

Start with some LHS

steps...
steps...
steps...

Show RHS

don't manipulate
LHS and RHS
at the same time.

[Example 1] Prove: *If* $a \mid b$ *and* $b \mid c$*, then* $a \mid c$*.*

"divides" means $\frac{b}{a}$ is an integer.

Can omit only for direct proofs with a single case.

proof: We use a direct proof. → method.

vars [Let $a, b, c \in \mathbb{Z}$.

assumptions [Assume $a \mid b$ and $b \mid c$.

This means $\exists\, m, n \in \mathbb{Z}$ such that $\frac{b}{a} = m$ and $\frac{c}{b} = n$.

massage [Then, $c = b \cdot n = (a \cdot m) \cdot n = a \cdot (m \cdot n) = a \cdot k$ for some $k \in \mathbb{Z}$.

conclusion [~~This becomes~~ Therefore, a divides c.

$\square$

[Example 2] **Prove:** *If n is an odd integer, then n^2+3n+5 is odd.*

hint: think about ways to represent even/odd numbers.

proof: We use a direct proof.

Let n be an odd integer, which means $n = 2k+1$ for some $k \in \mathbb{Z}$.

Then $n^2+3n+5 = (2k+1)^2 + 3(2k+1) + 5$

$$= 4k^2 + 4k + 1 + 6k + 3 + 5$$

$$= 4k^2 + 10k + 8 + 1$$

$$= 2(2k^2 + 5k + 4) + 1$$

$$= 2m + 1 \text{ for some } m \in \mathbb{Z}.$$

Therefore, n^2+3n+5 is odd. $\square$

Method #2: Split into cases (and massage).

e.g.:

$x < 0$? $x = 0$? $x > 0$?

method 1 method 2 method 3

direct/massage other method ... other method ..

[Example 3] **Prove:** *If $n \in \mathbb{N}$, then $1+(-1)^n(2n-1)$ is a multiple of 4.*

proof: We use a proof by cases where $n \in \mathbb{N}$ is either even or odd.

case 1: Let n be a positive even integer.

Then $n = 2k$ for some $k \in \mathbb{N}$.

always +1

This means $1+(-1)^n(2n-1) = 1 + (-1)^{2k}(2(2k)-1)$

$= 1 + (4k-1)$

$= 4k$ which is a multiple of 4.

case 2: Let n be a positive odd number.

Then $n = 2k+1$ for some $k \in \mathbb{N}$.

This means $1+(-1)^{2k+1}(2(2k+1)-1) = 1-(4k+2-1)$

$= -4k$ which is a multiple of 4.

Therefore, [expression] is a multiple of 4. $\square$