Learning objectives:

- ☐ create compound statements with $\wedge$, $\vee$ and $\neg$, and implications ($\implies$, $\iff$),
- ☐ identify the difference between a predicate and a proposition,
- ☐ prove compound statements using a truth table.

Last time, we talked about *propositions*, which are statements that are either true or false. The goal for today is to start deducing new truths from previous statements. To do that, we need to start with creating compound statements, so we need to introduce some symbols to make our notation easier. We'll also talk about *predicates*, which are similar to propositions, except that their truth depends on the value(s) of some variable(s). Before we define some symbols and predicates, let's warm up.

Did you know?

Last time, we talked about propositions, which are either *true* or *false*. Remember the name we use for variables like this in programming languages? Hint: they're named after George Boole (hence, `bool`). Did you know George Boole had a daughter who made big contributions to mathematics? Even though she never studied at a university, she is well known for her work on high-dimensional geometry. Apparently, she knew how to visualize the fourth dimension at a really young age! Her name is Alicia Boole Scott.

Example 1:

Say you invited four friends (Sulley, Mike Wazowski, Randall and Boo) over for pizza. You ordered a pizza with 12 slices. At the end of the night, you went to get one last slice, but there were none left! Who took the last slice? We must know. You asked your friends, and here's what they said:

- Sulley:
  - It wasn't Boo.
  - It wasn't Mike Wazowski.
- Mike Wazowski:
  - It wasn't Randall.
  - It was Boo.
- Randall:
  - It was Sulley.
  - It wasn't Boo.
- Boo:
  - It was Randall.
  - It wasn't Sulley.

We know that each of them told the truth in one of the statements and lied in the other. Can you tell who took the last slice of pizza?

Problem found here. Skip to the solution at the end of the lecture.

Example 2:

Which of the following are propositions?

1. The majority of Middlebury students are eco-friendly.
2. The product of 2 and 5.
3. $2 + x = 10$.

Solution:
Only (1) is a proposition. Sentence (2) has no verb and (3) depends on what x is.

1 Logical operators

Say we make the proposition p:

$$p = \textit{I am wearing a blue shirt},$$

along with a proposition q:

$$q = \textit{Today is Monday}.$$

We can combine p and q into a compound statement:

$$p \land q = \textit{I am wearing a blue shirt } \textbf{and} \textit{ today is Monday}.$$

Note the symbol $\land$ which is the **logical and** operator. We can also build statements like

$$p \lor q = \textit{I am wearing a blue shirt } \textbf{or} \textit{ today is Monday}.$$

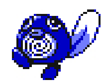
where we have now used the **logical or** ($\lor$) operator to join p and q. It's also useful to *negate* propositions, which we have an operator ($\neg$) for:

$$\neg p = \textit{I am } \textbf{not} \textit{ wearing a blue shirt.}$$

$$\neg q = \textit{Today is } \textbf{not} \textit{ Monday.}$$

Finally, we have the **logical xor** operator, which looks like $\oplus$. A statement $p \oplus q$ is true if *either* p or q are true, but not both.

Symbols!



Reading these symbols can be confusing at first (so try to practice). One way to remember that $\land$ means "and" is to think that it looks a lot like the letter "A" in the word "AND".

It's often useful to represent the results of these logic operators with a **truth table**.

p	q	$p \land q$	$p \lor q$	$p \oplus q$
T	T	T	T	F
T	F	F	T	T
F	T	F	T	T
F	F	F	F	F

1.1 Implications: if

You may have heard someone say a proposition such as: *If it is raining, I will carry an umbrella.* We have two propositions here: (1) *it is raining* and (2) *I will carry an umbrella*, which we are connecting with an implication. Whenever, you see an "if", it means an *implication*, which we represent with a single-sided arrow $\implies$. There is only one case in which the result of an implication is false:

- p is true but q is false.

See the truth table on the right.

Truth table for if ($\implies$)

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Perhaps a more intuitive way to think about an implication is with the following promise a professor might make to students: *If you get* 100% *on the final, you will get an A*. Overall, we want to evaluate the truth of what the professor promised. Now, if you get 100% on the final, and you get an A, then the promise is true (T $\implies$ T is T). However, you may also get an A in the course without getting 100% on the final (F $\implies$ T is T), which doesn't violate the professor's promise. Of course, if you don't get 100% on the final and don't get an A (F $\implies$ F is T), then the promise is also valid. However, if you get 100% on the final and don't get an A, then the promise would be invalid (T $\implies$ F is F).

1.2 if-and-only-if (iff)

In the raining-umbrella statement in the previous example, someone might still carry an umbrella even if it isn't raining, which seems pretty pointless. The statement: *I will carry an umbrella if-and-only-if it is raining* makes a little more sense. There's no point in carrying an umbrella if it isn't raining. We call this a **biconditional**, and represent it mathematically with a double-sided arrow, $\iff$. Anytime you see this, you should be thinking "iff"!

There are two cases in which the result of $p \iff q$ is true:

- p is true and q is true,
- p is false and q is false.

See the truth table on the right.

Truth table for iff ($\iff$)

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Does this mean that if 1+1 = 3, then I am a dinosaur?

Actually, yes. Both statements 1+1 = 3 and that you are a dinosaur are false, so the implication is true.

1.3 Necessary and sufficient

When reading a proof, you might encounter the terminology: *necessary* and *sufficient* conditions for a proposition to hold.

Definition 1. *Let p and q be statements:*

p is a ***sufficient condition*** *for q means* if p then q. *That is, if p holds, then this guarantees q also holds.*

p is a ***necessary condition*** *for q means* if not p then not q. *That is, if p does not hold, then there's no way that q holds. Note that we can also see this as* if q then p.

If p is a necessary condition for q, we can also see this as q being a sufficient condition for p. A ***necessary and sufficient*** *condition for a statement to hold is one in which with $(p \implies q) \land (q \implies p)$, which is the same as an* ***if-and-only-if*** *statement.*

Example 3:
Let p be the proposition *I am at least 18 years old* and let q be the proposition *I can vote*. Decide which of the following are true:

(a) p is a necessary condition for q

(b) p is a sufficient condition for q

(c) p is a necessary and sufficient condition for q

Solution:
p is a necessary and sufficient condition for q. Being at least 18 years old is a sufficient condition for voting. Also, not being 18 years old means that you cannot vote.

1.4 Truth table proofs

Truth tables can be used to prove compound propositions. In general, use these steps to do a truth table proof:

1. identify and label smaller propositions,
2. write the overall proposition symbolically,
3. create truth table and use it to show the proposition is always true.

Example 4:
Prove the following proposition is true. *If you eat spinach everyday, then you will win the lottery, or if you win the lottery, you will lose your job.*

Solution:
Let the statements be

- p: *you eat spinach everyday*.
- q: *you will win the lottery*.
- r: *you will lose your job*.

The full proposition is then $(p \implies q) \vee (q \implies r)$. The truth table is given below which has 2^3 rows for all possible truth values of the three statements p, q and r.

p	q	r	$p \implies q$	$q \implies r$	$(p \implies q) \vee (q \implies r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Since the proposition is always true (see the rightmost column), then we have proven the statement is true using a truth table.

In the last example, it was reasonable to prove the proposition using a truth table because there were only three variables. But what if you have hundreds of variables?? Or thousands?? In general, you will probably use a different technique. Consider the following argument as to whether the proposition is true.

Example 5:
Prove the proposition from the previous example without using a truth table.

Solution:
We know that q is true or it is false. If it is true, then $p \implies q$ is true. If it is false, then $q \implies r$ is true. Either way, at least one of $p \implies q$ and $q \implies r$ is true, so the whole statement is true.

2 de Morgan's laws

Sometimes, compound propositions can get a bit messy, and we would like to simplify them. Luckily we have a tool for that: *de Morgan's laws*. Given propositions p and q,

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

We also have a property of *double negation*

$$\neg\neg p \equiv p$$

and *distributive laws* for propositions p, q and r:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Be careful!

It might be tempting to just distribute a negation, but don't forget to flip the $\wedge$ and $\vee$. For example,

$$\neg(p \wedge q) \neq \neg p \wedge \neg q$$

Instead, we should flip the $\wedge$ with a $\vee$:

$$\neg(p \wedge q) = \neg p \vee \neg q$$

3 Predicates

So far, we have looked at propositions which are either true or false. This is kind of like creating a Boolean (`bool`) variable in a computer program. But sometimes, we need to create propositions that depend on the values of variables, or whether other propositions are true or false. These are called *predicates*.

Definition 2. *A* ***predicate*** *is a proposition whose truth depends on the value(s) of some variable(s).*

It's useful to include the variables that your predicate depends on in parentheses. For example:

$$p(x) = \text{Exit tickets are due on } x.$$

which is only true when the day of the week (x) is either Monday, Wednesday or Friday (see the table on the right).

x	$p(x)$
Monday	T
Tuesday	F
Wednesday	T
Thursday	F
Friday	T
Saturday	F
Sunday	F

Example 6:
Let $p(x)$ be the predicate $p(x) \equiv x > 0$. This is true when x is greater than zero (e.g. $p(5) = \text{T}$), but false otherwise (e.g. $p(0) = \text{F}$ and $p(-1) = \text{F}$).

We'll see more examples with predicates next class, when we look at sets and quantifiers.

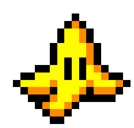
Example 7:
Check your understanding by trying to solve the logic puzzle from the beginning of the lecture with a truth table.

Solution:
We can write out whether each person ate the last slice in each row of a truth table, and then evaluate all eight statements. The subscripts refer to either of the two statements made by a particular person (S: Sulley, W: Mike Wazowski, R: Randall, B: Boo).

Ate last slice / Statement	S_1	S_2	W_1	W_2	R_1	R_2	B_1	B_2
Sulley (S)	T	T	T	F	T	T	F	F
Mike Wazowski (W)	T	F	T	F	F	T	F	T
Randall (R)	T	T	F	F	F	T	T	T
Boo (B)	F	T	T	T	F	F	F	T

Each person made one true statement and one false statement, so the only possibility is that the Mike Wazowski (W) took the last slice.

Careful!



One thing to note is that we said there was a pizza with 12 slices. Does the 12 matter here? Nope. This information wasn't necessary to solve the problem, but sometimes you're given extra (unnecessary) information, and you need to filter out what's important.