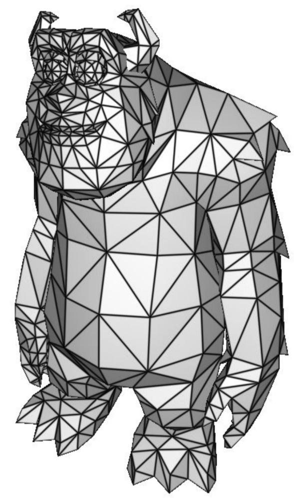
Learning objectives:

- ☐ build sets using both roster and builder notation
- ☐ perform operations on sets and subsets
- ☐ recall some famous sets: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{N}$

Last time, we talked about *predicates*, which were propositions whose truth depended on the value(s) some variable(s). But those variables need to *belong* to something. This is where *sets* come in.

Sets are extremely useful in discrete mathematics. In fact, when animation studios render movie characters, the character geometries are *discretized* into a mesh, which is a *set* of primitive shapes, like triangles, squares, etc.



Have a look at the picture on the right of Sulley from Monsters Inc - here, Sulley is represented by a set of triangle elements. The more triangle elements you have, the better you'll be able to represent the geometry of the character, but this might make the rendering pipeline slower.

Before we get into some symbols and terms, let's do a warm-up.

Example 1:
You decide to poll everyone in the class for what kind of chocolate they like. There are three chocolate options, which we will label as follows: milk (M), dark (D), ultra dark (U). In the poll, you allowed everyone to enter multiple choices. For example, someone might have entered either M, D, U, MD, MU, DU or MDU. The results of the poll are given below.

M	D	U	MD	MU	DU	MDU
7	9	2	3	2	1	1

The columns indicate the total number of people that like those kinds of chocolate. This means there is some overlap in the results. For example, there is overlap in the number of people who like M with MU MD and MDU. If you want to give one chocolate to everyone who said they like at least one type of chocolate, how many chocolates should you buy?

Click here to jump to the solution!

1 Terminology and symbols

Definition 1. *A **set** is a bunch of objects, called **elements**.*

These elements can be anything. In the case of Sulley, these elements are the triangles that, when combined, represent the geometry.

Here are some other examples of sets:

A	$= \{\text{dog}, \text{cat}, \text{bird}, \dots\}$	set of animals
B	$= \{2, 4, 6, 8, \dots\}$	set of even numbers
C	$= \{(0,0), (1,0), (2,0), \dots, (0,1), (1,1), (2,1), \dots\}$	set of points in the plane

The order in which the elements appear doesn't matter, and elements can appear only once in a set. We use the symbol $\in$ to represent that an element is in a set, and $\notin$ when an element is not in a set. For example, dog $\in A$, but table $\notin A$. Similarly, $2348 \in B$ but $5 \notin B$.

Here are some famous sets:

$$\varnothing = \{\} \qquad \text{(empty set)}$$

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \qquad \text{(set of integers)}$$

$$\mathbb{N} = \{1, 2, 3, \dots\} \qquad \text{(set of natural numbers)}$$

$$\mathbb{R} = \{\pi, \sqrt{2}, e, \dots\} \qquad \text{(set of real numbers)}$$

Sets can have an infinite number or a finite number of elements. We refer to the number of elements of a set as the *cardinality* of the set.

Definition 2. *The* ***cardinality*** *of a set S is the number of elements in the set, and is denoted by $|S|$.*

The sets $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ have infinite cardinality, whereas $\varnothing$ has cardinality 0. We also represent cardinality with two vertical bars around the variable representing the set. For example, $|\mathbb{Z}| = \infty$, whereas $|\varnothing| = 0$.

1.1 Describing sets

There are two ways to describe sets. The first is with **roster notation**, which explicitly lists out all elements of the set, enclosed within curly brackets $\{\}$. For example,

$$\text{weekdays} = \{\text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}, \text{Saturday}, \text{Sunday}\}$$

or

$$S = \{2, 4, 6, 8, 10\}$$

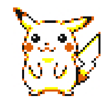
In the last example, S was a set containing the positive even numbers up to 10. But what if we want to list *all* even numbers?

A more useful way to describe sets is with **set-builder notation**, which makes use of a *predicate* to extract elements from a **domain**. Here is the general way to describe a set with set-builder notation:

$$S = \{x \mid p(x)\}$$

which means that S is a set made of elements x, such that $p(x)$ is true. These elements x can be taken from some larger set of elements X, $x \in X$, where X is called the **domain**. We can also extract elements from the domain by defining a function of x, $f(x)$. Note that the vertical bar means "such that".

Such that? "bar" versus "colon"



You might also see people use a "colon" instead of a vertical bar to symbolize "such that":

$$S = \{f(x) \;:\; p(x)\}\,.$$

Both mean the same thing.

Example 2:
Here are some examples using set-builder notation:

$$A = \{x \in \mathbb{Z} \mid x > 0, x \text{ is even}\}$$
$$B = \left\{x^2 + 2 \mid x \in \mathbb{R}\right\}$$
$$\mathbb{Q} = \left\{\frac{x}{y} \mid x, y \in \mathbb{Z}, y \neq 0\right\}$$

The last set, $\mathbb{Q}$, is actually the set of *rational numbers*.

Example 3:
Let $A = \{x^2 \mid \text{x is even}\}$. Can you think of another way to express A?

Solution:
Here are some options:

- $A = \{x \mid \frac{\sqrt{x}}{2} \in \mathbb{Z}\}$
- $A = \{x^2 \mid \frac{x}{2} \in \mathbb{Z}\}$
- $A = \{(2x)^2 \mid x \in \mathbb{Z}\}$

1.2 Cartesian product

We can also build sets that are ordered pairs of elements in two sets. Actually, you've been doing this for a while now! Think about the Cartesian plane. It consists of all the ordered pairs (x, y) in which x and y are elements of $\mathbb{R}$. We have a special way of representing this, using the $\times$ symbol.

Definition 3. *The* ***Cartesian product*** *of two sets A and B is the set of* ordered pairs *of elements in A with those in B:*

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Note that the order matters! For example, we can write the Cartesian plane as $\mathbb{R} \times \mathbb{R}$. Cartesian products come up in a lot of places. For example, a deck of cards can be expressed as the Cartesian product of the sets $\{2, \ldots, 10, J, K, Q, A\}$ with $\{H, D, S, C\}$, referring to hearts (H), diamonds (D), spades (S) and clubs (S). You can also take the Cartesian product of a set with itself.

Example 4:
Let $H(x,y)$ be the predicate that x has said hello to y. Let S be the set of all students in cs200. Which of the following can be used to express the set of student introductions?

(a) $\{x \in S \mid y \in S, H(x,y)\}$

(b) $\{(x,y) \in S \mid H(x,y)\}$

(c) $\{(x,y) \in S \times S \mid H(x,y)\}$

(d) $S \times S$

Solution:
Option (a) doesn't account for the *pair* of students forming the introduction. Option (b) isn't mathematically correct because pairs are not elements of the set of students S. Option (d) consists of all possible student interactions but doesn't account for whether they have actually said hello and introduced themselves. Option (c) is the only correct way of expressing the introductions.

More symbols!

Think of $\subseteq$ as the analog of the "$\leq$" symbol and the $\subset$ as the analog of of "$<$".

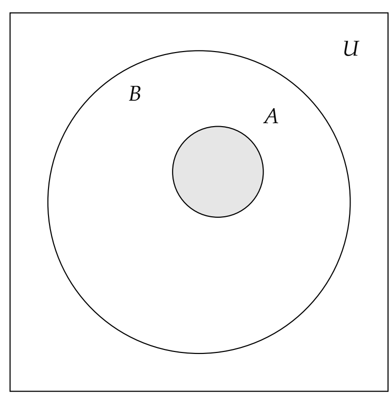
1.3 Subsets

We can also extract a portion of the elements of a set, to form a *subset*.

Definition 4. *Let B be a set.*

- *A set A is a* ***subset*** *of B, denoted by $A \subseteq B$ if every element of A is also in B.*
- *A set A is a* ***strict subset*** *of B, denoted by $A \subset B$, if every element of A is also in B, but $A \neq B$.*

It's sometimes useful to visualize subsets with a Venn diagram, which will be particularly useful later in the course when we discuss *counting*. In the figure on the right, I've also included the **universal set** U (see the square), which is a set that contains all the values other sets can be formed from. For example, we might have $U = \mathbb{Z}$ (all integers), $B = \mathbb{Z}^+$ (positive integers) and $A = \{x \mid x = 2y,\ y \in B\}$ (positive even integers).



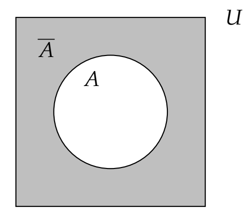
1.4 Set operations

Let's now define some *operations* on sets. Let A and B be sets which are subsets of some universal set U.

Complement

Definition 5. *The **complement** of a set A, denoted by $\overline{A}$ is defined by*

$$\overline{A} = U - A = \{x \in U \mid x \notin A\}.$$

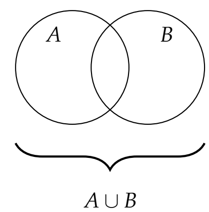


That is, the complement consists of all elements in the universe U that are not in A.

Union The union of two sets A and B consists of all elements that are in *either* A or B.

Definition 6. *The **union** of a set A and a set B, denoted by $A \cup B$, is defined by*

$$A \cup B = \{x \mid x \in A \lor x \in B\}.$$



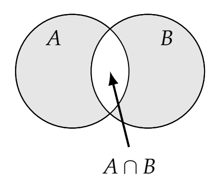
For example,

$$\{a, b\} \cup \{c, d, a\} = \{a, b, c, d\}$$

Intersection The intersection of two sets A and B consists of all elements that are in *both* A and B.

Definition 7. *The **intersection** of a set A and a set B, denoted by $A \cap B$, is defined by*

$$A \cap B = \{x \mid x \in A \land x \in B\}.$$



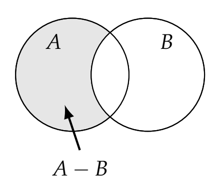
For example,

$$\{a, b\} \cap \{c, d, a\} = \{a\}$$

Difference The difference between two sets A and B consists of all elements that are in A, but not in B.

Definition 8. *The **difference** between a set A and a set B, denoted by $A - B$, is defined by*

$$A - B = \{x \mid x \in A \land x \notin B\}.$$



Let us now revisit the problem from the beginning of the lecture and solve it using sets and cardinalities.

Example 5:

Solution:

We know that 7 people like milk chocolate (M), $|M| = 7$. We also know that $|D| = 9$ and $|U| = 2$. However, we also know that $|M \cap D| = 3$ since we said that the number of people who marked down that they like both milk *and* dark chocolate is 2. Conversely, $|M \cap U| = 2$ and $|D \cap U| = 1$. There is only one person who likes all three types: $|M \cap D \cap U| = 1$. To determine how many people like chocolate, we need to determine how many people responded to the poll. We start by adding the cardinalities of the sets M, D and U. However, this would overcount each of the categories since those who marked down M may also have voted for D. So we need to subtract out the intersections: $|M \cap D|$, $|M \cap U|$ and $|D \cap U|$. But wait! We wouldn't be accounting for the single person who voted for all three! So we need to add back in $|M \cap D \cap U|$.

$$\begin{aligned} |M \cup D \cup U| &= |M| + |D| + |U| - |M \cap D| - |M \cap U| - |D \cap U| + |M \cap D \cap U| \\ &= 7 + 9 + 2 - 3 - 2 - 1 + 1 = 13 \end{aligned}$$

So we need to buy 13 chocolates so that everyone who wants a chocolate, receives one. See if you can draw the Venn diagram to visualize the cardinality of $|M \cup D \cup U|$.

Don't worry

Don't worry too much if this example isn't clear right now. This is known as the *principle of inclusion/exclusion* which we will study later in the course when we discuss *counting* techniques.